Limit of a function of real variable
Chapter-3

1.**Limit point of a set:** Let $D \subset \mathbf{R}$, a real number c is said to be a **limit point** of D if every nbd of c (say $(c-\delta, c+\delta)$, $\delta > 0$ a real number) contains infinitely many elements of D. The set of all such limit point of D is denoted by D' and is known as the derive set of D. The set D' may have no element, or only a finite number of elements or may have infinite number of elements.

2. **Bounded Function:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function, f is said to be
(i). **bounded above** on D if $\exists$ a real number B s.t. $f(x) \leq B \; \forall \; x \in D$, B is said to be the upper bound of f on D.
(ii). **bounded below** on D if $\exists$ a real number b s.t. $f(x) \geq b \; \forall \; x \in D$, b is said to be the lower bound of f on D.
(iii)**bounded** on D if it is bounded above as well as bounded below on D. Clearly f is bounded on D if the range set $f(D)$ is a bounded subset of $\mathbf{R}$.

3. **Supremum (l.u.b and infimum (g.l.b)of $f(x)$)**
Let f be bounded above on D. Then the range set $f(D) = \{f(x) : x \in D\}$ is a nonempty bounded above subset of $\mathbf{R}$. So by the supremum property of $\mathbf{R}$, the subset $f(D)$ has a least upper bound M(say). M is called the **supremum** of f on D and is expressed as $M = Sup \; f(x)$. Clearly the supremum M of f satisfies the following conditions
(i). $f(x) \leq M \; \forall \; x \in D$ and
(ii). for each pre-assigned $\epsilon > 0 \; \exists$ a $y \in D$ s.t. $M - \epsilon < f(y) \leq M$
Let f be bounded below on D. Then the range set $f(D) = \{f(x) : x \in D\}$ is a nonempty bounded below subset of $\mathbf{R}$. Hence the subset $f(D)$ has a greatest lower bound m(say). m is called the **infimum** of f on D and is expressed as $m = inf \; f(x)$. Clearly the infimum m of f satisfies the following conditions
(i). $m \leq f(x) \; \forall \; x \in D$ and
(ii). for each pre-assigned $\epsilon > 0 \; \exists$ a $y \in D$ s.t. $m < f(y) \leq m + \epsilon$
Note that f is said to be unbounded if it is either unbounded below or unbounded above or both.
If $f : D \to \mathbf{R}$ be a bounded function on D, then $[sup \; f(x) - inf \; f(x)]$is called the oscillation of f on D

4. **Bounded function at a point:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function and $c \in D$. f is said to be bounded at c,

if $\exists$ a nbd $N(c)$ of c s.t. $f$is bounded on $N(c) \cap D$.
Note:- Give an example of a function $f : D \to \mathbf{R}$ s.t. f is bounded at each point of D without being bounded on D.
Ans: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$, Clearly it is bounded at each point of $\mathbf{R}$, but not bounded on $\mathbf{R}$.

5. **Increasing and Decreasing functions (Monotone function:-):**

Def^n-1:A function f defined on an interval I is said to be **increasing** in I if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-2:A function f defined on an interval I is said to be **strictly increasing** in I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-3:A function f defined on an interval I is said to be **decreasing** in I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-4:A function f defined on an interval I is said to be **strictly decreasing** in I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-5:A function is known as **monotone** in an interval I if it is either increasing or decreasing in I.
Def^n-6:A function is known as **strictly monotone** in an interval I if it is either strictly increasing or strictly decreasing in I.

6.**Limit of the function at a point:**

Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let c is a limit point of D. A real number l is said to be the **limit of the function** $f(x)$ as $x \to c$, written as $lim_{x \to c} f(x) = l$ if given any $\epsilon > 0$, $\exists$ a positive real number δ such that $|f(x) - l| < \epsilon \quad \forall x \in \quad (c - \delta, c + \delta) - \{c\}$, i.e., there exists a deleted nbd of c in which the difference between the functional values $\forall x \in \quad (c - \delta, c + \delta) - \{c\}$ and the real number l can be made however small.
Since $c \in D' \Rightarrow$ c is a limit point of D. So it may or may not belongs to D, even if $c \in D$, $f(c)$ need not lie in the ϵ -nbd of l.

7. **The limit of a function, if exists is unique.**

Proof:- Let $D \subset \mathbf{R}$ and c is a limit point of D. If possible let $f : D \to \mathbf{R}$ be such that $lim_{x \to c} f(x) = l$. and $lim_{x \to c} f(x) = m$, where $l \neq m$. Now the possible relations among the two real numbers l and m are either $l < m$ or $l > m$. So let $l < m$ and $\epsilon = \frac{m-l}{2} > 0$, then $l + \epsilon = m - \epsilon$. Therefore the open intervals $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are clearly disjoint.
Now $lim_{x \to c} f(x) = l \;\; \Rightarrow l - \epsilon < f(x) < l + \epsilon \;\; \forall \; x \in (c - \delta_1, c + \delta_1) - \{c\}$ for some real number $\delta_1 > 0$.
Again $lim_{x \to c} f(x) = m \;\; \Rightarrow m - \epsilon < f(x) < m + \epsilon \;\; \forall \; x \in (c - \delta_2, c + \delta_2) - \{c\}$ for

some real number $\delta_2 > 0$.
Let $\delta = \min\ \{\delta_1, \delta_2\}$, then both the above two inequalities are simultaneously true for all $x \in (c-\delta, c+\delta) - \{c\}$, which is a contradiction to the fact that the open intervals $(l-\epsilon, l+\epsilon)$ and $(m-\epsilon, m+\epsilon)$ are disjoint. Hence the relation $l < m$ is impossible.
Similarly it can be shown easily that the relation $l > m$ is also impossible.
Hence the theorem.

8.**Sequential Criterion on limit:**
Statement:- Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$ and $l \in \mathbf{R}$. Then $\lim_{x\to c} f(x) = l$ iff for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.
Proof:- Let $\lim_{x\to c} f(x) = l$. Then for given $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t.
$l - \epsilon < f(x) < l + \epsilon\ \forall\ x \in N'(c, \delta) \cap D$.............(1)
Let $\{x_n\}$ sequence in $D - \{c\}$ converging to c,i.e. $lim\ x_n = c$. So, given any $\delta > 0$ $\exists$ $k \in \mathbf{N}$ s.t. $c - \delta < x_n < c + \delta\ \forall n \geq k$.................(2)
(1) and (2) together gives $l - \epsilon < f(x_n) < l + \epsilon\ \forall n \geq k$.............(3) i.e. $lim f(x_n) = l$ and hence the sequence $\{f(x_n)\}$ converges to l.
Conversely, let for every sequence in $D - \{c\}$ converging to c, $lim f(x_n) = l$, then we have to prove that $\lim_{x\to c} f(x) = l$
If not, let $\exists$ a nbd V of l s.t. for every nbd W of c $\exists$ at least one element $x_w \in (W - \{c\}) \cap D$ for which $f(x_w)$ does not belongs to V.
Let $W_1 = N(c, 1)$, then $\exists$ an element $x_1 \in N'(c, 1) \cap D$ s.t.$f(x_1) \notin V$.
Let $W_2 = N(c, \frac{1}{2})$, then $\exists$ an element $x_2 \in N'(c, \frac{1}{2}) \cap D$ s.t.$f(x_2) \notin V$.
Proceeding in this manner, we obtain a sequence $\{x_1, x_2, ...\}$ in D s.t. $lim\ x_n = c$, since $x_n \in W_n = N(c, \frac{1}{n})\ \forall n \in \mathbf{N}$, but the sequence $\{f(x_n)\}$ does not converge to l. Since $f(x_n) \notin$ nbd V of $l\ \forall n \in \mathbf{N}$. This is a contradiction to the hypothesis and therefore $\lim_{x\to c} f(x) = l$. Hence the result.

9. **Statement:-**Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$. If f has a limit $l \in \mathbf{R}$ at c then f is bounded on $N(c)$ of c. Is the converse true, justify.
Proof;- Since $\lim_{x\to c} f(x) = l$, where $c \in D'$
So for $\epsilon = 1$ $\exists$ a $\delta > 0$ s.t. $|f(x) - l| < 1\ \forall\ x \in N'(c, \delta) \cap D$
Now $||f(x)| - |l|| < |f(x) - l|$. So, $||f(x)| - |l|| < 1\ \forall\ x \in N'(c, \delta) \cap D$
$\Rightarrow |f(x)| < 1 + |l|\ \forall\ x \in N'(c, \delta) \cap D$
$\Rightarrow f(x)$ is bounded on $N(c, \delta) \cap D$
The converse of the theorem is not true i.e. if f is not bounded on $N(c, \delta) \cap D$ for some $\delta-$nbd $N(c, \delta)$ of c then $\lim_{x\to c} f(x)$ does not exists in $\mathbf{R}$.

For example, let us consider the limit, $\lim_{x\to 0}\frac{1}{x}$. Let, $f(x)=\frac{1}{x},\ x\in D$. Here $D=\mathbf{R}-\{0\}$ and $0\ \in\ D'$, f is unbounded on every nbd of zero. Therefore $f(x)=\frac{1}{x},\ x\in D$ does not exists in $\mathbf{R}$.

10. **Statement:** Let $D\subset\mathbf{R}$ and $f: D\to\mathbf{R}$ be a function. Let $c\ \in D'$ and $\lim_{x\to c}f(x)=l$.
(i)If $l>\ 0$ then $\exists$ a $\delta\ >0$ s.t. $f(x)\ >0\forall x\ \in\ N'(c,\delta)\cap\ D$
(ii)If $l<\ 0$ then $\exists$ a $\delta\ >0$ s.t. $f(x)\ <0\forall x\ \in\ N'(c,\delta)\cap\ D$
Proof: (i). Since $l>0$, so for a suitable $\epsilon\ >0$, we have $l-\epsilon\ >0$. Again since $\lim_{x\to c}f(x)=l$. So for the same $\epsilon>0$ $\exists$a $\delta>0$ s.t. $l-\epsilon\ <f(x)\ <l+\epsilon\ \ \forall x\ \in\ N'(c,\delta)\cap\ D\Rightarrow\ 0\ <l-\epsilon<f(x)\forall x\ \in\ N'(c,\delta)\cap\ D\Rightarrow f(x)\ >0\ \forall x\ \in\ N'(c,\delta)\cap\ D$.
(ii) Since $l<0$, so for a suitable $\epsilon\ >0$, we have $l+\epsilon\ <0$. Again since $\lim_{x\to c}f(x)=l$. So for the same $\epsilon>0$ $\exists$a $\delta>0$ s.t. $l-\epsilon\ <f(x)\ <l+\epsilon\ \ \forall x\ \in\ N'(c,\delta)\cap\ D\Rightarrow f(x)<l+\epsilon<0\ \forall x\ \in\ N'(c,\delta)\cap\ D\Rightarrow f(x)\ <0\forall x\ \in\ N'(c,\delta)\cap\ D$.
Hence the result.

11. **Algebra of limits:** Let $D\subset\mathbf{R}$ and $f,g: D\to\mathbf{R}$ are two functions on D. Let $c\in D'$ and $\lim_{x\to c}f(x)=l$, $\lim_{x\to c}g(x)=m$, then
(i)$lim_{x\to\ c}(f+g)(x)=l+m$.
(ii)$lim_{x\to\ c}(kf)(x)=kl$, where $k\in\ \mathbf{R}$.
(iii) $lim_{x\to\ c}(f.g)(x)=lm$.
(iv)Moreover if $g(x)\ \neq\ 0\forall\ x\in\ D$ and $m\ \neq 0$ then $lim_{x\to c}\frac{f}{g}(x)=\frac{l}{m}$
Proof:(i) Since $|(f+g)(x)-(l+m)|=|(f(x)-l)+(g(x)-m)|\ \leq\ |f(x)-l|+|g(x)-m|$...........(1)
Again since, $\lim_{x\to c}\ f(x)=l$, $\lim_{x\to c}g(x)=m$, so for any given $\epsilon>0$ $\exists\delta_1>0$ and $\delta_2\ >\ 0$ s.t.
$|f(x)-l|\ <\ \frac{\epsilon}{2}\ \forall x\ \in\ N'(c,\delta_1)\cap\ D$ and $|g(x)-m|\ <\ \frac{\epsilon}{2}\ \forall x\ \in\ N'(c,\delta_2)\cap\ D$. Let $\delta=min(\delta_1,\delta_2)$. Then from above we have, for $\forall x\ \in\ N'(c,\delta)\cap\ D$ the relations $|f(x)-l|\ <\ \frac{\epsilon}{2}$ and $|g(x)-m|\ <\ \frac{\epsilon}{2}$ are simultaneously satisfied. Hence (1) becomes $|(f+g)(x)-(l+m)|<(\frac{\epsilon}{2}+\frac{\epsilon}{2})=\epsilon\ \forall x\ \in\ N'(c,\delta)\cap\ D$. Hence the result.
(ii) Since, $\lim_{x\to c}\ f(x)=l$, so for a given $\epsilon\ >0\ \exists\ \delta\ >0$ s.t. $|f(x)-l|<\frac{\epsilon}{|k|}\ \forall\ x\in\ N'(c,\delta)\cap D$. Thus $|kf(x)-kl|=|k||f(x)-l|<|k|\frac{\epsilon}{|k|}=\epsilon\ \forall\ x\in\ N'(c,\delta)\cap D$. Hence the result.
(iii) Since,$|fg(x)-lm|=|(f(x)-l)g(x)+l(g(x)-m)|\leq\ |f(x)-l||g(x)|+|l||g(x)-m|$. Again since, $\lim_{x\to c}g(x)$ exists, so $g(x)$ is bounded in the nbd of c and therefore there exists a positive number B and a positive number $\delta_3>0$ s.t.
$|f(x)|<B\ \forall x\ \in\ N'(c,\delta_1)\cap\ D$.

Finally, since $\lim_{x\to c} f(x) = l$, $\lim_{x\to c} g(x) = m$, so for any given $\epsilon > 0$ $\exists\delta_1 > 0$ and $\delta_2 > 0$ s.t.
$|f(x) - l| < \frac{\epsilon}{2B}$ $\forall x \in N'(c,\delta_1)\cap D$ and $|g(x) - m| < \frac{\epsilon}{2|l|}$ $\forall x \in N'(c,\delta_2)\cap D$.
Let $\delta = min(\delta_1,\delta_2,\delta_3)$. Then from above we have, $|fg(x) - lm| < B\frac{\epsilon}{2B} + |l|\frac{\epsilon}{2|l|} = \epsilon$ $\forall x \in N'(c,\delta)\cap D$. Hence the result.
(iv)Since $lim_{x\to c}\frac{f}{g}(x) = lim_{x\to c}f(x)\frac{1}{g(x)}$. So to prove $lim_{x\to c}\frac{f}{g}(x) = \frac{l}{m}$, keeping the result obtained in (iii) in our mind (you have to write the proof of result (iii) for a separate question)it sufficient to prove only the result $lim_{x\to c}\frac{1}{g(x)} = \frac{1}{m}$. Now since, $|\frac{1}{g(x)} - \frac{1}{m}| = \frac{|g(x)-m|}{|m||g(x)|}$. Now since $\lim_{x\to c} g(x) = m$, so corresponding to given $\epsilon_1 = \frac{1}{2}|m|$ $\exists$ $\delta_1 > 0$ s.t. $|m| - \frac{1}{2}|m| < |g(x)| < |m| + \frac{1}{2}|m|$ $\forall x \in N'(c,\delta_1)\cap D \Rightarrow$ $\frac{1}{2}|m| < |g(x)| < \frac{3}{2}|m|$ $\forall x \in N'(c,\delta_1)\cap D$
Again, $\lim_{x\to c} g(x) = m$, so corresponding to given $\epsilon > 0$ $\exists$ a $\delta_2 > 0$ s.t.
$|g(x) - m| < \frac{\epsilon\ |m|^2}{2}$ $\forall x \in N'(c,\delta_2)\cap D$
Let $\delta = min(\delta_1,\delta_2)$, then from above we have $|\frac{1}{g(x)} - \frac{1}{m}| < \epsilon$ $\forall$ $x \in N'(c,\delta)\cap D$.
Hence the result.

12.**Statement:-**Let $D \subset \mathbf{R}$ and $f, g : D \to \mathbf{R}$ are two functions from D to $\mathbf{R}$. Let $c \in D'$. If f is bounded on $N'(c)\cap D$ for some deleted nbd $N'(c)$ of c and $\lim_{x\to c} g(x) = 0$ then show that $\lim_{x\to c} f(x)g(x) = 0$
Proof:Since $f$is bounded on $N'(c)\cap D$ for some nbd $N(c)$ of c. So there exists a positive number B and a positive number δ_1 s.t. $|f(x)| < B\forall\ x \in N'(c,\delta_1)\cap D$. Again since $\lim_{x\to c} g(x) = 0 \Rightarrow$ Given any $\epsilon > 0$ $\exists$ $\exists_2 > 0$ s.t. $|g(x) - 0| < \frac{\epsilon}{B}\forall\ x \in N'(c,\delta_2)\cap D$. Let $\delta = min(\delta_1,\delta_2)$ then from above we have, $|f(x)| < B$ and $|g(x) - 0| < \frac{\epsilon}{B}$ $\forall\ x \in N'(c,\delta)\cap D$, thus $|fg(x-0)| = |f(x)||g(x)| < \epsilon$ $\forall\ x \in N'(c,\delta)\cap D$. Hence the result.

13.**Statement:**Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a functions on D. Let $c \in D'$, if $f(x) \le b$ $\forall$ $x \in D - \{c\}$ and if $lim_{x\to c}f(x) = l$ then show that $l \le b$.
Proof: Let $\{x_n\}$ be any sequence in $D - \{c\}$ converging to c. Since,$lim_{x\to c}f(x) = l$, therefore by sequential criterion $lim_{x\to c}f(x_n) = l$. Let us consider a constant sequence $\{u_n\}$ s.t. $u_n = b\forall\ n \in \mathbf{N}$, then $lim\ u_n = b$. Since $f(x) \le b$ $\forall$ $x \in D - \{c\}$
Hence $f(x_n) \le u_n\forall n \in \mathbf{N} \Rightarrow l \le b$.

14. **Sandwich theorem:**Let $D \subset \mathbf{R}$ and $f, g, h : D \to \mathbf{R}$ are functions from D to $\mathbf{R}$. Let $c \in D'$ and $f(x) \le g(x) \le h(x)\forall\ x \in D - \{c\}$. If $lim_{x\to c}f(x) = lim_{x\to c}h(x) = l$ then prove that $lim_{x\to c}g(x) = l$.
Proof: Let $\epsilon > 0$ be arbitrary and $\delta_i, i = 1, 2, 3$. As $lim_{x\to c}f(x) = lim_{x\to c}h(x) = l$

and $f(x) \leq g(x) \leq h(x) \forall x \in D - \{c\}$ we have the followings
$|f(x) - l| < \epsilon \, \forall \, x \in N'(c, \delta_1) \cap D$
$|h(x) - l| < \epsilon \, \forall \, x \in N'(c, \delta_2) \cap D$ and
$f(x) \leq g(x) \leq h(x) \forall \, x \in N'(c, \delta_3) \cap D$
Thus if $\delta = min(\delta_i, i = 1, 2, 3)$ then we have
$l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon \, \forall \, x \in N'(c, \delta) \cap D$
$\Rightarrow l - \epsilon < g(x) < l + \epsilon \, \forall \, x \in N'(c, \delta) \cap D \Rightarrow lim_{x \to c} g(x) = l.$

15. **Cauchy's principal on limit:**(statement only)
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$. A necessary and sufficient condition for the existence of $lim_{x \to c} f(x) = l$ is that for a pre-assigned $\epsilon > 0 \, \exists \, \delta > 0$ s.t. $|f(x') - f(x'')| < \epsilon \, \forall$ pair of points $x', x'' \in N'(c, \delta) \cap D$.

16. **Right hand limit and Left hand limit:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $D_1 = \{x \in D : x > c\}$ then $c \in D_1'$. Now a real number l is said to be the R.H.L of $f(x)$ at c if given any $\epsilon > 0 \, \exists \, \delta > 0$ s.t. $|f(x) - l| < \epsilon \, \forall x \in N'(c, \delta) \cap D_1$ i.e. $\forall \, x \in (c, c + \delta)$. Then we write $lim_{x \to c^+} f(x) = l$.

Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $D_2 = \{x \in D : x < c\}$ then $c \in D_2'$. Now a real number l is said to be the L.H.L of $f(x)$ at c if given any $\epsilon > 0 \, \exists \, \delta > 0$ s.t. $|f(x) - l| < \epsilon \, \forall x \in N'(c, \delta) \cap D_2$ i.e. $\forall \, x \in (c - \delta, c)$. Then we write $lim_{x \to c^-} f(x) = l$.

Sequential criterion: Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $D_1 = \{x \in D : x > c\}$ then $c \in D_1'$. Now $lim_{x \to c^+} f(x) = l$ iff for every sequence $\{x_n\}$ in D_1 converges to c, the sequence $\{f(x_n)\}$ converges to l.
Similarly let $D_2 = \{x \in D : x < c\}$ then $c \in D_2'$. Now $lim_{x \to c^-} f(x) = l$ iff for every sequence $\{x_n\}$ in D_2 converges to c, the sequence $\{f(x_n)\}$ converges to l.

17. **Statement:-**Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $D_1 = \{x \in D : x > c\}$ and $D_2 = \{x \in D : x < c\}$. So c is a limit point of both D_1 and D_2, then prove that a real number l is said to be the limit of $f(x)$ as $x \to c, i.e. \, lim_{x \to c} f(x) = l$ iff $lim_{x \to c^+} f(x) = l = lim_{x \to c^-} f(x)$.
Proof: Let $lim_{x \to c} f(x) = l$ then given any $\epsilon > 0 \, \exists \, \delta > 0$ s.t. $|f(x) - l| < \epsilon \, \forall \, x \in N'(c, \delta) \cap D \Rightarrow |f(x) - l| < \epsilon$ whenever $0 < |x - c| < \delta$. So $|f(x) - l| < \epsilon \, \forall \, x \in (c, c + \delta) \cap D$ and $|f(x) - l| < \epsilon \, \forall \, x \in (c - \delta, c) \cap D$. i.e. $|f(x) - l| < \epsilon \, \forall \, x \in N'(c, \delta) \cap D_1$ and $|f(x) - l| < \epsilon \, \forall \, x \in N'(c, \delta) \cap D_2$. Hence $lim_{x \to c^+} f(x) = l = lim_{x \to c^-} f(x)$.
Conversely, let $lim_{x \to c^+} f(x) = l = lim_{x \to c^-} f(x)$. Then given any $\epsilon > 0 \, \exists \, \delta_i >$

$0, i=1,2$ s.t.$|f(x)-l| < \epsilon \; \forall \; x \; \in N'(c,\delta) \cap D_1$ and $|f(x)-l| < \epsilon \; \forall \; x \; \in N'(c,\delta) \cap D_2$. Let $\delta = min(\delta_i, i=1,2)$ then from above we have $|f(x)-l| < \epsilon \; \forall \; x \; \in N'(c,\delta) \cap D \Rightarrow \; lim_{x\to \; c} f(x) = l$.

18.**Infinite limits:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$. If corresponding to a pre-assigned (+)ve number $G \; \exists$ a positive number δ s.t. $f(x) > G \; \forall x \in \; N'(c,\delta) \cap D$, then we say that f tend to ∞ as $x \to c$ and we write $lim_{x\to c} f(x) = \infty$.

Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$. If corresponding to a pre-assigned (+)ve number $G \; \exists$ a positive number δ s.t. $f(x) < -G \; \forall x \in \; N'(c,\delta) \cap D$, then we say that f tend to $-\infty$ as $x \to c$ and we write $lim_{x\to c} f(x) = -\infty$.

Sequential criterion: Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $c \in D'$. Then $lim_{x\to c} f(x) = \infty$ iff for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c the sequence $\{f(x_n)\}$ diverges to ∞
Similarly, $lim_{x\to c} f(x) = -\infty$ iff for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c the sequence $\{f(x_n)\}$ diverges to $-\infty$

19.**Limit at infinity:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(c,\infty) \subset \; D$ for some $c \in \; \mathbf{R}$. Then $lim_{x\to\infty} f(x) = l \; \in \; \mathbf{R}$ if corresponding to $\epsilon \; > 0 \; \exists$ a real number $G \; > c$ s.t. $|f(x)-l| < \epsilon \; \forall \; x > \; G$
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(-\infty,c) \subset \; D$ for some $c \in \; \mathbf{R}$. Then $lim_{x\to-\infty} f(x) = l \; \in \; \mathbf{R}$ if corresponding to $\epsilon \; > 0 \; \exists$ a real number $G \; < c$ s.t. $|f(x)-l| < \epsilon \; \forall \; x < \; G$

Sequential criterion: Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(c,\infty) \subset \; D$ for some $c \in \; \mathbf{R}$. Then $lim_{x\to\infty} f(x) = l \; \in \; \mathbf{R}$ iff for every sequence $\{x_n\}$ in (c,∞) diverges to ∞, the sequence $\{f(x_n)\}$ converges to l.
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(-\infty,c) \subset \; D$ for some $c \in \; \mathbf{R}$. Then $lim_{x\to-\infty} f(x) = l \; \in \; \mathbf{R}$ iff for every sequence $\{x_n\}$ in $(-\infty,c)$ diverges to $-\infty$, the sequence $\{f(x_n)\}$ converges to l.

20. **Infinite limit at infinity:**
Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(c,\infty) \subset \; D$ for some $c \in \; \mathbf{R}$. Then $lim_{x\to\infty} f(x) = \infty (or, -\infty)$ if corresponding to a pre-assigned (+)ve number $G \; \exists$ a real number $k > c$ s.t. $f(x) \; > G(or, < -G) \forall \; x \; > k$.

Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbf{R}$. Then $lim_{x\to-\infty} f(x) = \infty(or, -\infty)$ if corresponding to a pre-assigned (+)ve number G $\exists$ a real number $k < c$ s.t. $f(x) > G(or, < -G) \forall\ x < k$

Sequential criterion: Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbf{R}$. Then $lim_{x\to\infty} f(x) = \infty(or, -\infty)$ iff for every sequence $\{x_n\}$ in (c, ∞) diverges to ∞, the sequence $\{f(x_n)\}$ diverges to $\infty(or, -\infty)$.

Let $D \subset \mathbf{R}$ and $f : D \to \mathbf{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbf{R}$. Then $lim_{x\to-\infty} f(x) = \infty(or, -\infty)$ iff for every sequence $\{x_n\}$ in (c, ∞) diverges to $-\infty$, the sequence $\{f(x_n)\}$ diverges to $\infty(or, -\infty)$.

21. **Statement:-** let $f : D \to \mathbf{R}$ be a function and $(c, \infty) \subset D$ for some $c \in \mathbf{R}$. Then $lim_{x\to\infty} f(x) = l \in \mathbf{R}$ iff $lim_{x\to 0^+} f(\frac{1}{x}) = l$

Proof: Let $g(x) = \frac{1}{x}$ then $lim_{x\to 0^+} g(x) = \infty$. Since $lim_{x\to\infty} f(x) = l$. So for given $\epsilon > 0 \ \exists\ d > 0$ s.t. $|f(x) - l| < \epsilon\ \forall x > d$. Again since $lim_{x\to 0^+} g(x) = \infty$, so the chosen (+)ve number $d\ \exists\ \delta > 0$ s.t. $g(x) > d\ \forall x \in (0, \delta)$. Thus $|fg(x) - l| < \epsilon\ \forall\ x \in (0, d) \Rightarrow lim_{x\to 0^+} fg(x) = l$ $i.e.$ $lim_{x\to 0^+} f(\frac{1}{x}) = l$

Conversely let, $lim_{x\to 0^+} fg(x) = l$, then for pre-assigned $\epsilon > 0\ \exists\ \delta > 0$ s.t. $l - \epsilon < fg(x) < l + \epsilon\ \forall\ x \in (0, \delta) \cap D$, D being the domain of fg. Now $x \in (0, \delta) \Rightarrow g(x) > \frac{1}{\delta}$. Hence $l - \epsilon < f(x) < l + \epsilon\ \forall\ x > \frac{1}{\delta} \Rightarrow lim_{x\to\infty} f(x) = l$. Hence the result.

22A.**Important Theorems (statement only:)**

Let $f : I \to \mathbf{R}$ be a monotone increasing function on any bounded open interval $I = (a, b)$,

(i). if f is bounded above on I, then $lim_{x\to b^-} f(x) = Sup_{x\in(a,b)}\ f(x)$

(ii). if f is bounded below on I, then $lim_{x\to a^+} f(x) = inf_{x\in(a,b)}\ f(x)$

(iii). if f is unbounded above on I, then $lim_{x\to b^-} f(x) = \infty$

(iv). if f is unbounded below on I, then $lim_{x\to a^+} f(x) = inf_{x\in(a,b)}\ -\infty$

22B.**Important Theorems (statement only:)**

Let $f : I \to \mathbf{R}$ be a monotone decreasing function on any bounded open interval $I = (a, b)$,

(i). if f is bounded above on I, then $lim_{x\to a^+} f(x) = Sup_{x\in(a,b)}\ f(x)$

(ii). if f is bounded below on I, then $lim_{x\to b^-} f(x) = inf_{x\in(a,b)}\ f(x)$

(iii). if f is unbounded above on I, then $lim_{x\to a^+} f(x) = \infty$

(iv). if f is unbounded below on I, then $lim_{x\to b^-} f(x) = inf_{x\in(a,b)}\ -\infty$

23.**Statement:**Let $f : I \to \mathbf{R}$ be a monotone function on any bounded open interval $I = (a, b)$ and $c \in (a, b)$, then show that $lim_{x\to c^-} f(x)$ and $lim_{x\to c^+} f(x)$ both exist.

Proof: Clearly the given function may be monotone increasing or monotone decreasing and accordingly we have the following two cases

Case-1: Let $f : I \rightarrow \mathbf{R}$ be a monotone increasing function on the bounded open interval $I = (a, b)$ and $c \in (a, b)$.

Then f is m.i. and bounded above on (a, c) and $f(c)$ is an upper bound of f on (a, c). Let M be the supremum of f on (a, c), then $lim_{x \to c^-} f(x) = M \leq f(c)$.

Again, f is m.i. and bounded below on (c, b) and $f(c)$ is a lower bound of f on (c, b). Let m be the infimum of f on (c, b), then $f(c) \leq m = lim_{x \to c^+} f(x)$.

Thus $lim_{x \to c^-} f(x)$ and $lim_{x \to c^+} f(x)$ both exist and $lim_{x \to c^-} f(x) \leq lim_{x \to c^+} f(x)$.

Case-2: Let $f : I \rightarrow \mathbf{R}$ be a monotone decreasing function on the bounded open interval $I = (a, b)$ and $c \in (a, b)$.

Then f is m.d. and bounded below on (a, c) and $f(c)$ is an lower bound of f on (a, c). Let m be the infimum of f on (a, c), then $f(c) \leq lim_{x \to c^-} f(x) = m$.

Again, f is m.d. and bounded above on (c, b) and $f(c)$ is an upper bound of f on (c, b). Let M be the supremum of f on (c, b), then $M = lim_{x \to c^+} f(x) \leq f(c)$.

Thus $lim_{x \to c^-} f(x)$ and $lim_{x \to c^+} f(x)$ both exist and $lim_{x \to c^+} f(x) \leq lim_{x \to c^-} f(x)$.