1 Expansion of functions

1.1 Rolle's Theorem

Statement: *Let f be a real valued function of the real variable x defined in the closed interval [a,b] be such that*
(i)it is continuous in the closed interval [a,b]
(ii) it is derivable in the open interval (a,b) and
(iii) $f(a) = f(b)$
then $\exists$ *at least one value of x, say* $\xi \in$ *(a,b) s.t* $f^{'}(\xi) = 0$
Proof: The function f, being continuous in the closed interval [a,b], is bounded therein and attains its least upper (M)and greatest lower (m)bound (*lub* and *glb* respectively). Then $\exists$ at least, one ξ, and one $\eta \in$ [a,b] be such that $f(\xi) = M$ and $f(\eta) = m$. Now M, m are two reals, so either $M = m$ or $M \neq m$.
If $M = m$, then $f(\xi) = M$ $\forall$ x $\in$ [a,b] $\Rightarrow$ $f'(\xi) = 0$ $\forall$ x $\in$ [a,b] and the theorem is obvious.
Let us suppose that $M \neq m$, then with the condition (iii)of the statement, we can say at least one of M and m must be different from $f(a)$ and $f(b)$.
Let $f(a) = f(b) \neq M \Rightarrow f(a) = f(b) \neq f(\xi) \Rightarrow f(a) \neq f(\xi)$ and $f(b) \neq f(\xi) \Rightarrow$ $\xi \neq a$ and $\xi \neq b$ but $\xi \in$ [a,b], hence $\xi \in$ (a,b).
Thus by the remaining condition (ii) of the statement we can say $f'(\xi)$ exists. Now it remains to show $f'(\xi) = 0$, if not let $f'(\xi) > 0$. Now $f'(\xi) > 0 \Rightarrow \exists$ a real number $\delta > 0$ s.t $f(x) > f(\xi) = M$ $\forall$ x $\in (\xi, \xi + \delta)$, which is a contradiction to the fact that $M = lub$ of f(x)in [a,b]. So we can not have $f'(\xi) > 0$.
Again if possible let $f'(\xi) < 0$, then depending on the similar discussion as above we have $f(x) > f(\xi) = M$ $\forall$ x $\in (\xi - \delta, \xi)$, which is again a contradiction to the fact that $M = lub$ of f(x)in [a,b]. So we can not have $f'(\xi) < 0$.
Hence we can conclude that the only possibility is $f'(\xi) = 0$.

Geometrical interpretation of Rolle's theorem:
If the graph of y=f(x) has equal ordinate at the two points A,B and if the graph be continuous throughout the interval from A to B and if the curve has a tangent at every point on it from A to B except possibly at the two points A and B, then there must exist at least one point on the curve between A and B, where the tangent is parallel to x-axis.

1.2 Lagrange's Mean Value Theorem (MVT)or First MVT of Differential calculus:

Statement: *Let f be a real valued function of the real variable x defined in the closed interval [a,b] be such that*
(i)it is continuous in the closed interval [a,b]
(ii) it is derivable in the open interval (a,b)
then $\exists$ *at least one value of x, say* $\xi \in$ *(a,b) s.t*

$$f(b) - f(a) = (b - a)f'(\xi), \qquad \textit{where } \xi \in (a, b)$$

Proof:Let us consider a real valued function $\phi(x)$ of the real variable x defined in the closed interval [a,b] as follows

$$\phi(x) = f(x) + Ax, \qquad \text{where A is a constant such that } \phi(a) = \phi(b).$$

Then we have

$$f(a) + Aa = f(b) + A.b$$
$$\Rightarrow A = -\frac{f(b) - f(a)}{b - a} \tag{1}$$

Clearly with the help of given conditions and with the restriction on A, we can say
(i)$\phi(x)$ is continuous in the closed interval [a,b]
(ii) $\phi(x)$ is derivable in the open interval (a,b) and
(iii) $\phi(a) = \phi(b)$
Thus all the conditions of Rolle's theorem are met by the function $\phi(x)$. Hence $\exists$ at least one value of x, say $\xi \in$ (a,b) s.t $\phi^{'}(\xi) = 0$

$$\Rightarrow A = -f'(\xi) \tag{2}$$

Thus Eq.(1) and Eq.(2) together gives

$$f(b) - f(a) = (b - a)f'(\xi), \qquad \text{where } \xi \in (a, b)$$

Geometrical interpretation of Lagrange's MVT
Let A and B be two points on the graph of y=f(x) corresponding to $x = a$ and $x = b$ respectively. Then the slope of the chord AB $=\frac{f(b)-f(a)}{b-a}$ and we know that geometrically $f'(\xi)$ represent the gradient of the tangent at the point $x = \xi$ on the curve. Thus MVT geometrically indicated the fact that $\exists$ at least one point $\xi \in (a, b)$, the tangent at which on the curve is parallel to the chord joining the points $x = a$ and $x = b$.

1.3 Increasing and Decreasing functions:

Def^n-1:A function f defined on an interval I is said to be **increasing** in I if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-2:A function f defined on an interval I is said to be **strictly increasing** in I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-3:A function f defined on an interval I is said to be **decreasing** in I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-4:A function f defined on an interval I is said to be **strictly decreasing** in I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.
Def^n-5:A function is known as **monotone** in an interval I if it is either increasing or decreasing in I.
Def^n-6:A function is known as **strictly monotone** in an interval I if it is either strictly increasing or strictly decreasing in I.
Theorem-1:*If a real valued function f of the real variable x defined in the closed interval [a,b] be such that*
(i) it is continuous in the closed interval [a,b]
(ii) it is derivable in the open interval (a,b) and
(iii) $f'(x) = 0 \forall x \in (a, b)$
then show that $f(x)$ *has a* ***constant value*** *throughout [a,b].*
Proof: Let $c \in (a, b)$ be arbitrary. Then according to the given conditions
(i)f is continuous in the closed interval [a,c]
(ii) f is derivable in the open interval (a,c)
Thus f satisfies all the conditions of Lagrange's MVT on [a,c] and hence $\exists$ a real number $d \in (a, c)$ s.t

$$f(c) - f(a) = (c - a)f'(d), \qquad \text{where } d \in (a, c)$$

Thus with the help of condition (iii) of the theorem we have

$$f'(d) = 0, \qquad \text{as } d \in (a, c) \subset (a, b)$$

Hence from above, we have $f(c) = f(a)$. But c is arbitrary, so $f(x) = f(a)$ $\forall x \in [a, c)$. Similarly it can be shown that $f(x)$ satisfies all the conditions of Lagrange's MVT in (c,b] and by similar manner we can conclude that $f(c) = f(b)$ and hence $f(x) = f(a) = f(b) \forall x \in [a, b]$. Hence the proof is completed.
Theorem-2:*If f(x)and g(x) are both defined on [a,b] such that*
(i)Both are continuous in the closed interval [a,b]
(ii) Both are derivable in the open interval (a,b), and

(iii) $f'(x) = g'(x) \forall x \in (a, b)$ *then show that f(x)and g(x)are differ by a constant on [a,b]*

Proof: Let $\phi(x) = f(x) - g(x)$, then the given conditions of this theorem help us to say

(i) $\phi(x)$ is continuous in the closed interval [a,b]

(ii) $\phi(x)$ is derivable in the open interval (a,b) and

(iii) $\phi'(x) = 0 \ \ \forall x \in (a, b)$

Thus all the conditions of the previous theorem (Theorem-1)are satisfied by the function $\phi(x)$ and by the conclusion of the previous theorem (Theorem-1)we have $\phi(x)$ =constant $\forall x \in [a, b]$. Thus f(x)and g(x)are differ by a constant on [a,b].

Theorem-3:*If f is continuous on [a,b] and* $f'(x) \geq 0$ *in (a,b) then show that f is increasing in[a,b].*

Proof: Let x_1 and x_2 be any two distinct points of [a,b] such that $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Hence $\exists$ a number $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

Since $f'(x) \geq 0 \forall x \in (a, b)$. Thus $f'(c) \geq 0$ as $c \in (x_1, x_2) \subset (a, b)$ and $x_2 - x_1 > 0$. Hence from above we have $f(x_2) - f(x_1) \geq 0$, $\Rightarrow f(x_2) \geq f(x_1)$, whenever $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. So $f(x)$ is increasing in [a,b].

Theorem-4:*If f is continuous on [a,b] and* $f'(x) > 0$ *in (a,b) then show that f is strictly increasing in[a,b].*

Proof: Let x_1 and x_2 be any two distinct points of [a,b] such that $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Hence $\exists$ a number $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

Since $f'(x) > 0 \forall x \in (a, b)$. Thus $f'(c) > 0$ as $c \in (x_1, x_2) \subset (a, b)$ and $x_2 - x_1 > 0$. Hence from above we have $f(x_2) - f(x_1) > 0$, $\Rightarrow f(x_2) > f(x_1)$, whenever $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. So $f(x)$ is strictly increasing in [a,b].

Theorem-5:*If f is continuous on [a,b] and* $f'(x) \leq 0$ *in (a,b) then show that f is decreasing in[a,b].*

Proof: Let x_1 and x_2 be any two distinct points of [a,b] such that $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Hence $\exists$ a number $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

Since $f'(x) \leq 0 \forall x \in (a, b)$. Thus $f'(c) \leq 0$ as $c \in (x_1, x_2) \subset (a, b)$ and $x_2 - x_1 > 0$. Hence from above we have $f(x_2) - f(x_1) \leq 0$, $\Rightarrow f(x_2) \leq f(x_1)$, whenever $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. So $f(x)$ is decreasing in [a,b].

Theorem-6: *If f is continuous on [a,b] and $f'(x) < 0$ in (a,b) then show that f is strictly decreasing in[a,b].*

Proof: Let x_1 and x_2 be any two distinct points of [a,b] such that $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Hence $\exists$ a number $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

Since $f'(x) < 0 \forall x \in (a, b)$. Thus $f'(c) < 0$ as $c \in (x_1, x_2) \subset (a, b)$ and $x_2 - x_1 > 0$. Hence from above we have $f(x_2) - f(x_1) < 0$, $\Rightarrow f(x_2) < f(x_1)$, whenever $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. So $f(x)$ is strictly decreasing in [a,b].

Theorem-7: *If $f'(x)$ exists and is bounded on some interval I, then f is uniformly continuous on I.*

Proof:Since $f'(x)$ is bounded on the interval I, therefore $\exists$ a real number $k > 0$ s.t

$$|f'(x)| \leq k \quad \forall x \in I,$$

et x_1 and x_2 be any two distinct points of I such that $x_1 < x_2$. Then $[x_1, x_2] \subset I$ and f satisfies all the conditions of Lagrange's mean value theorem in $[x_1, x_2]$. Hence $\exists$ a number $c \in (x_1, x_2)$ such that

$$\tfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c),$$

$$\Rightarrow \tfrac{|f(x_2)-f(x_1)|}{|x_2-x_1|} \leq k \quad (since |f'(x)| \leq k \quad \forall x \in I).$$

$$\Rightarrow |f(x_2) - f(x_1)| \leq k|x_2 - x_1| \quad \forall x_1, x_2 \in I.$$

Let $\epsilon > 0$ be given, then if we choose $\delta = \frac{\epsilon}{k}$ we have

$$|f(x_2) - f(x_1)| < \epsilon \quad \text{whenever } |x_2 - x_1| < \delta.$$

Hence f is uniformly continuous on I.

1.4 Cauchy's Mean Value Theorem (MVT)or Second MVT of Differential calculus:

Statement: *Let f and g be two real valued functions of the real variable x defined in the closed interval [a,b] be such that*
(i)Both are continuous in the closed interval [a,b]
(ii) Both are derivable in the open interval (a,b), and
(iii)$g'(x) \neq 0 \ \forall x \in (a,b)$
then $\exists$ *at least one value of x, say* $\xi \in$ *(a,b) s.t*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}, \qquad \textit{where } \xi \in (a,b)$$

Proof:Let us consider a real valued function $\phi(x)$ of the real variable x defined in the closed interval [a,b] as follows

$$\phi(x) = f(x) + Ag(x), \qquad \text{where A is a constant such that } \phi(a) = \phi(b).$$

Then we have

$$\begin{aligned} f(a) + Ag(a) &= f(b) + Ag(b) \\ \Rightarrow A &= -\frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned} \tag{3}$$

Provided $g(b) \neq g(a)$ and it is so because of the fact that if $g(b) = g(a)$, then all the conditions of Rolle's theorem are met by the function $g(x)$ in [a,b] and then $\exists$ at least one $\xi \in (a,b)$ s.t $g'(\xi) = 0$, which contradict the condition (iii) of the statement.
Now it is clear that the function $\phi(x)$ is
(i) continuous in [a,b] (because of the condition (i) of the statement of the theorem)
(ii) derivable in (a,b) (because of the condition (ii) of the statement of the theorem), and
(iii)$\phi(a) = \phi(b)$ (according to the restriction on ϕ).
Thus all the conditions of Rolle's theorem are met by the function $\phi(x)$. Hence $\exists$ at least one value of x, say $\xi \in$ (a,b) s.t $\phi^{'}(\xi) = 0$

$$\Rightarrow A = -\frac{f'(\xi)}{g'(\xi)}, \qquad g'(\xi) \neq 0 \tag{4}$$

Thus Eq.(3) and Eq.(4) together gives

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}, \qquad \text{where } \xi \in (a,b)$$

Note: It is important to mention here that whenever $g(x) = x$, the cauchy's MVT will transformed to the Lagrange's MVT, hence Lagrange's MVT is a particular case of cauchy's MVT.

Geometrical interpretation of Cauchy's MVT

The tangents at the points $(\xi, f(\xi))$ and $(\xi, g(\xi))$ on the graph of the functions $f(x)$ and $g(x)$ are mutually parallel.

1.5 Taylor's Theorem with Lagrange's form of Remainder :

Statement:*If a real valued function f of a real variable x defined in the closed interval [a,a+h] be such that*
(i)the (n-1)th derivative f^{n-1} *is continuous in the closed interval [a,a+h]*
(ii) the n-th derivative f^n *exists in the open interval (a,a+h)*
then $\exists$*at least one number* $\theta \in (0,1)$ *such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \text{.........} + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h)$$

The last term $R_n = \frac{h^n}{n!}f^n(a+\theta h)$ is called the Lagrange's form of remainder after n terms.

Proof: Let us consider a function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \text{.........} + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a+h-x)^n A$$

Where the constant A is so chosen that $\phi(a) = \phi(a+h)$

$$\Rightarrow f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \text{.........} + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h^n A = f(a+h) \qquad (5)$$

Clearly with the help of given conditions and with the restriction on A, we can say
(i)$\phi(x)$ is continuous in the closed interval [a,b], (since $f.f', f'', ...f^{n-1}$ are all continuous in [a,a+h]and $(a+h-x)^i$, i=1,2,3,...n are everywhere continuous.)
(ii) $\phi(x)$ is derivable in the open interval (a,b)(since $f.f', f'', ...f^{n-1}$ are all derivable in (a,a+h)and $(a+h-x)^i$, i=1,2,3,...n are always derivable.) and
(iii) $\phi(a) = \phi(b)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(x)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(a+\theta h) = 0$, But since

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + ... + \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) - \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n}(x) - n(a+h-x)^{n-1}A$$
$$\Rightarrow \phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n}(x) - n(a+h-x)^{n-1}A$$

Hence $\phi'(a+\theta h) = 0 \Rightarrow A = \frac{f^n(a+\theta h)}{n!}$, then putting this value of A in Eq.(5) we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h), \text{ where } 0 < \theta < 1$$

1.6 Taylor's Theorem with Cauchy's form of Remainder :

Statement:*If a real valued function f of a real variable x defined in the closed interval [a,a+h] be such that*
(i)the (n-1)th derivative f^{n-1} is continuous in the closed interval [a,a+h]
(ii) the n-th derivative f^n exists in the open interval (a,a+h)
then $\exists$at least one number $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$$

The last term $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$ is called the Cauchy's form of remainder after n terms.

Proof:Let us consider a function $\phi(x)$ defined by
$\phi(x) =$

$$f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + ... + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a+h-x)A$$

Where the constant A is so chosen that $\phi(a) = \phi(a+h)$

$$\Rightarrow f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + hA = f(a+h) \qquad (6)$$

Clearly with the help of given conditions and with the restriction on A, we can say
(i)$\phi(x)$ is continuous in the closed interval [a,b], (since $f.f', f'', ...f^{n-1}$ are all continuous in [a,a+h]and $(a+h-x)^i$, i=1,2,3,...(n-1) are everywhere continuous.)
(ii) $\phi(x)$ is derivable in the open interval (a,b)(since $f.f', f'', ...f^{n-1}$ are all derivable in (a,a+h)and $(a+h-x)^i$, i=1,2,3,...(n-1) are always derivable.) and
(iii) $\phi(a) = \phi(b)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(x)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(a+\theta h) = 0$, But since

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + ... + \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) - \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n}(x) - A$$

$$\Rightarrow \phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n}(x) - A$$

Hence $\phi'(a+\theta h) = 0 \Rightarrow A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{n}(a+\theta h)$, then putting this value of A in Eq.(6) we get

$f(a+h) =$

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^{n}(1-\theta)^{n-1}}{(n-1)!}f^{n}(a+\theta h), \text{ where } 0 < \theta < 1$$

1.7 Taylor's Theorem with Generalized form of Remainder :

Statement:*If a real valued function f of a real variable x defined in the closed interval [a,a+h] be such that*
(i)the (n-1)th derivative f^{n-1} is continuous in the closed interval [a,a+h]
(ii) the n-th derivative f^{n} exists in the open interval (a,a+h) and
(iii)p is any given positive integer.
then $\exists$at least one number $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^{n}(a+\theta h)$$

The last term $R_n = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^{n}(a+\theta h)$ is called the Schlömilch-Röche's form of remainder after n terms.

Proof:Let us consider a function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + ... + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a+h-x)^{p}A,$$

Where p is a positive integer and the constant A is so chosen that $\phi(a) = \phi(a+h)$

$$\Rightarrow f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h^{p}A = f(a+h) \qquad (7)$$

Clearly with the help of given conditions and with the restriction on A, we can say (i)$\phi(x)$ is continuous in the closed interval [a,b], (since $f.f', f'', ...f^{n-1}$ are all continuous in [a,a+h]and $(a+h-x)^{i}$, i=1,2,3,...(n-1),p are everywhere continuous.)

(ii) $\phi(x)$ is derivable in the open interval (a,b)(since $f.f', f'', ...f^{n-1}$ are all derivable in (a,a+h)and $(a+h-x)^i$, i=1,2,3,...(n-1),p are always derivable.) and
(iii) $\phi(a) = \phi(b)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(x)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(a + \theta h) = 0$, But since

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + ... + \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) - \frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - Ap(a+h-x)^{p-1}$$
$$\Rightarrow \phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - Ap(a+h-x)^{p-1}$$

Hence $\phi'(a + \theta h) = 0 \Rightarrow A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^n(a + \theta h)$, then putting this value of A in Eq.(7) we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h),$$
where $0 < \theta < 1$

1.8 Maclaurin's Theorem with Lagrange's form of Remainder :

Statement:*If a real valued function f of a real variable t defined in the closed interval [0,x] be such that*
(i)the (n-1)th derivative f^{n-1} is continuous in the closed interval [0,x]
(ii) the n-th derivative f^n exists in the open interval (0,x)
then $\exists$at least one number $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$$

The last term $R_n = \frac{x^n}{n!}f^n(\theta x)$ is called the Lagrange's form of remainder after n terms.
Proof: Let us consider a function $\phi(t)$ defined by

$$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + ... + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + (x-t)^n A$$

Where the constant A is so chosen that $\phi(0) = \phi(x)$

$$\Rightarrow f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + x^n A = f(x) \qquad (8)$$

Clearly with the help of given conditions and with the restriction on A, we can say
(i)$\phi(t)$ is continuous in the closed interval [0,x], (since $f.f', f'', ...f^{n-1}$ are all continuous in [0,x]and $(x-t)^i$, i=1,2,3,...n are everywhere continuous.)
(ii) $\phi(t)$ is derivable in the open interval (0,x)(since $f.f', f'', ...f^{n-1}$ are all derivable in (0,x)and $(x-t)^i$, i=1,2,3,...n are always derivable.) and
(iii) $\phi(0) = \phi(x)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(t)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(\theta x) = 0$, But since

$$\phi'(t) = f'(t) - f'(t) + (x-t)f''(t) - (x-t)f''(t) + ... + \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) - \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^{n}(t) - n(x-t)^{n-1}A$$
$$\Rightarrow \phi'(t) = \frac{(x-t)^{n-1}}{(n-1)!}f^{n}(t) - n(x-t)^{n-1}A$$

Hence $\phi'(\theta x) = 0 \Rightarrow A = \frac{f^n(\theta x)}{n!}$, then putting this value of A in Eq.(8) we get

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$, where $0 < \theta < 1$

1.9 Maclaurin's Theorem with Cauchy's form of Remainder :

Statement:*If a real valued function f of a real variable t defined in the closed interval [0,x] be such that*
(i)the (n-1)th derivative f^{n-1} is continuous in the closed interval [0,x]
(ii) the n-th derivative f^n exists in the open interval (0,x)
then $\exists$at least one number $\theta \in (0,1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$$

The last term $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(x\theta)$ is called the Cauchy's form of remainder after n terms.
Proof:Let us consider a function $\phi(t)$ defined by
$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + ... + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + (x-t)A$ Where the constant A is so chosen that $\phi(0) = \phi(x)$

$$\Rightarrow f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + xA = f(x) \quad (9)$$

Clearly with the help of given conditions and with the restriction on A, we can say (i)$\phi(t)$ is continuous in the closed interval [0,x], (since $f.f', f'', ...f^{n-1}$ are all continuous in [0,x]and $(x-t)^i$, i=1,2,3,...(n-1) are everywhere continuous.)
(ii) $\phi(t)$ is derivable in the open interval (0,x)(since $f.f', f'', ...f^{n-1}$ are all derivable in (a,a+h)and $(x-t)^i$, i=1,2,3,...(n-1) are always derivable.) and
(iii) $\phi(0) = \phi(x)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(t)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(\theta x) = 0$, But since

$$\phi'(t) = f'(t) - f'(t) + (x-t)f''(t) - (x-t)f''(t) + ... + \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) - \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - A$$
$$\Rightarrow \phi'(t) = \frac{(x-t)^{n-1}}{(n-1)!}f^n(t) - A$$

Hence $\phi'(\theta x) = 0 \Rightarrow A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$, then putting this value of A in Eq.(9) we get
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$, where $0 < \theta < 1$

1.10 Maclaurin's Theorem with Generalized form of Remainder :

Statement:*If a real valued function f of a real variable t defined in the closed interval [0,x] be such that*
(i)the (n-1)th derivative f^{n-1} is continuous in the closed interval [0,x]
(ii) the n-th derivative f^n exists in the open interval (0,x) and
(iii)p is any given positive integer.
then $\exists$at least one number $\theta \in (0,1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$$

The last term $R_n = \frac{x^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$ is called the Schlömilch-Röche's form of remainder after n terms.
Proof:Let us consider a function $\phi(t)$ defined by

$$\phi(t) = f(t) + (x-t)f'(t) + \frac{(x-t)^2}{2!}f''(t) + ... + \frac{(x-t)^{n-1}}{(n-1)!}f^{n-1}(t) + (x-t)^p A,$$

Where p is a positive integer and the constant A is so chosen that $\phi(0) = \phi(x)$

$$\Rightarrow f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + x^p A = f(x) \qquad (10)$$

Clearly with the help of given conditions and with the restriction on A, we can say (i)$\phi(t)$ is continuous in the closed interval [0,x], (since $f.f', f'', ...f^{n-1}$ are all continuous in [0,x]and $(x-t)^i$, i=1,2,3,...(n-1),p are everywhere continuous.)
(ii) $\phi(t)$ is derivable in the open interval (0,x)(since $f.f', f'', ...f^{n-1}$ are all derivable in (0,x)and $(x-t)^i$, i=1,2,3,...(n-1), p are always derivable.) and
(iii) $\phi(0) = \phi(x)$ (by construction)
Thus all the conditions of Rolle's theorem are met by the function $\phi(t)$. Hence $\exists$ at least one number θ with $0 < \theta < 1$ such that $\phi'(\theta x) = 0$, But since

$$\phi'(t) = f'(t) - f'(t) + (x-t)f''(t) - (x-t)f''(t) + ... + \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) - \frac{(x-t)^{n-2}}{(n-2)!}f^{n-1}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^{n}(t) - Ap(x-t)^{p-1}$$
$$\Rightarrow \phi'(t) = \frac{(x-t)^{n-1}}{(n-1)!}f^{n}(t) - Ap(x-t)^{p-1}$$

Hence $\phi'(\theta x) = 0 \Rightarrow A = \frac{x^{n-p}(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$, then putting this value of A in Eq.(10) we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x), \text{ where } 0 < \theta < 1$$

2 Important Problems on this Chapter

2.1 Short answer type questions

2.1.1 For what value of p the function $f(x)$ =cos x -2px is monotonically decreasing. BU2001,

Ans:- The given function will be monotonically decreasing if $f'(x) \leq 0 \Rightarrow sinx + 2p \geq 0 \Rightarrow 1 \geq sinx \geq -2p \quad \Rightarrow p \geq -\frac{1}{2}$

2.1.2 Show that $log_e(1+x) < x - \frac{x^2}{2(1+x)}$ for $x > 0$. BU2001

Ans:- Let us consider the function $f(x) = x - \frac{x^2}{2(1+x)} - log_e(1+x)$.
Then clearly f(x)is continuous for all $x > 0$ and $f'(x) = 1 - \frac{2x+2x^2-x^2}{2(1+x)^2} - \frac{1}{1+x} = \frac{2+4x+2x^2-2x-x^2-2-2x}{2(1+x)^2} = \frac{x^2}{2(1+x)^2} > 0 \ \ \forall x > 0$. So $f(x)$ is increasing for all $x > 0$ and hence $f(x) > f(0) \ \ \forall x > 0 \Rightarrow \ \ log_e(1+x) < x - \frac{x^2}{2(1+x)}$

2.1.3 Show that the functions $f(x) = tanx$ is monotone increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$. BU2002

Ans:- Here $f'(x) = sec^2x > 0 \forall -\frac{\pi}{2} < x < \frac{\pi}{2}$. Hence the result.

2.1.4 Examine applicability of Rolle's theorem in [-1,1]for functions (i) $f(x) = |x|$ and (ii) $g(x) = x^3$ BU2002,2008

Ans:- (i)Since $f(x)$ is not differential at $x = 0$ and hence the Rolle's theorem is not applicable for the given function in the given interval.
(ii) Clearly although being a polynomial function $g(x)$ is continuous in [-1,1]and differentiable in $(-1, 1)$ as $f(-1) = -1$ and $f(1) = 1$, so $f(-1) \neq f(1)$. So the Rolle's theorem is not applicable for the given function $g(x)$ also in the give interval.

2.1.5 Give an example of a function which satisfies all the condition of Rolle's theorem, where the derivative of the function vanishes at two interior points. BU2003

Ans:- Let us consider the function $f(x) = sinx$ in $[-\pi, \pi]$, Clearly this function is continuous in $[-\pi, \pi]$ and it is differentiable in $(-\pi, \pi)$ and $f(-\pi) = 0 = f(\pi)$. Thus all the three conditions of Rolle's theorem are satisfied for the considered function in the considered interval. Clearly $f'(x) = cosx$ and $f'(-\frac{\pi}{2}) = 0 = f'(\frac{\pi}{2})$, where $-\frac{\pi}{2}, \ \ \frac{\pi}{2} \in [-\pi, \pi]$. Hence the result.

2.1.6 Is the function $f(x) = cosx$ monotone in $(-\frac{\pi}{2}, \frac{\pi}{2})$, justify your answer. BU2003

Ans:-Here $f'(x) = -sinx$, so $f'(x) > 0 \forall x \in (-\frac{\pi}{2}, 0)$ and $f'(x) < 0 \forall x \in (0, \frac{\pi}{2})$. Hence $f(x) = cosx$ is monotone increasing $\forall x \in (-\frac{\pi}{2}, 0)$ and monotone decreasing $\forall x \in (0, \frac{\pi}{2})$.

2.1.7 Show that between any two real roots of $e^x sinx = 1$ there is a real root of $1 + e^x cosx = 0$. BU2004,2006

Let α and β be two roots of the equation $f(x) = e^{-x} - sinx = 0$. Then clearly $f(\alpha) = 0 = f(\beta)$, and $f(x)$is continuous in (α, β). Further $f'(x) = -e^{-x} - cosx$ exists in (α, β). Thus all the three conditions of Rolle's theorem are satisfied for the function $f(x)$ in (α, β). Thus by the conclusion of the theorem we can say $\exists$ a value of $x = \gamma$in (α, β) such that $f'(\gamma) = 0$. Thus between any two real roots of $f(x) = e^{-x} - sinx = 0$ i.e., of $e^x sinx = 1$ there is a real root of $f'(x) = -e^{-x} - cosx = 0$ i.e., of $1 + e^x cosx = 0$.

2.1.8 Use Rolle's theorem to show that between any two real roots of $e^x sinx = 1$ there is a real root of $e^x(sinx + cosx) = 0$. BU2009,

Do yourself

2.1.9 For $x > 0$ prove that $log_e(1 + x) > x - \frac{x^2}{2}$. BU2004

Let us consider the function $f(x) = x - \frac{x^2}{2} - log_e(1 + x)$. Then clearly $f(x)$ is continuous for all $x > 0$ and $f'(x) = 1 - x - \frac{1}{1+x} = -\frac{x^2}{1+x} < 0 \quad \forall x > 0$. So $f(x)$ is decreasing for all $x > 0$ and hence $f(x) < f(0) \quad \forall x > 0 \Rightarrow \quad log_e(1 + x) > x - \frac{x^2}{2}$.

2.1.10 Show that the functions $f(x) = tanx$ is monotone increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$. BU2002

Ans:- Here $f'(x) = sec^2x > 0 \forall -\frac{\pi}{2} < x < \frac{\pi}{2}$. Hence the result.

2.1.11 Can you apply Rolle's theorem to $f(x) = 1 - |x|$ in [-1,1]? Give reasons for answer. BU2005

Ans:-Clearly the given function is not differentiable at $x = 0 \in (-1, 1)$,i.e., $f(x) = 1 - |x|$ is not differentiable in $(-1, 1)$, and hence the Rolle's theorem is not applicable to the given function in the given interval.

2.1.12 With proper justification show that $sinx < x < tanx$ for $0 < x < \frac{\pi}{2}$, BU2006

Ans:- Let us consider the function $f(x) = x - sinx$. Then $f'(x) = 1 - cosx > 0 \quad \forall x \in (0, \frac{\pi}{2})$. Therefore $f(x)$ is a monotone increasing function in $(0, \frac{\pi}{2})$. Hence $f(x) > f(0) \quad \forall x > 0 \in (0, \frac{\pi}{2}) \quad \Rightarrow sinx < x$ for $0 < x < \frac{\pi}{2}$.
Again let us consider the function $g(x) = tanx - x$. Then $g'(x) = sec^2x - 1 =$

$tan^2x > 0 \ \ \forall x \in (0, \frac{\pi}{2})$. Therefore $g(x)$ is a monotone increasing function in $(0, \frac{\pi}{2})$. Hence $g(x) > g(0) \ \ \forall x > 0 \in (0, \frac{\pi}{2}) \quad \Rightarrow x < tanx$ for $0 < x < \frac{\pi}{2}$.
Now combining the above two results we get the desired results.

2.1.13 Show that $\frac{2x}{\pi} < sinx < x$ in $0 < x < \frac{\pi}{2}$. BU 2006

Ans:- Let us consider the function $f(x) = \frac{sinx}{x}$ when $x \neq 0$ and $f(0) = 1$. Then $f(x)$ is continuous in $[0, \frac{\pi}{2}]$ and differentiable in $(0, \frac{\pi}{2})$ and $f'(x) = \frac{xcosx - sinx}{x^2}$. Again let us consider another function $g(x) = xcosx - sinx$ in $[0, \frac{\pi}{2}]$. Clearly $g'(x) = -xsinx < 0 \ \ \forall \ \ 0 < x < \frac{\pi}{2}$. Hence $g(x)$ is strictly decreasing in $0 \leq x \leq \frac{\pi}{2}$. Hence $g(x) < g(0) = 0 \ \ in \ \ 0 \leq x \leq \frac{\pi}{2}$. Hence $f'(x) < 0$ for $0 \leq x \leq \frac{\pi}{2}$ and hence $f(x)$ is strictly decreasing in $0 \leq x \leq \frac{\pi}{2}$. So $f(\frac{\pi}{2}) < f(x) < f(0)$ for $0 \leq x \leq \frac{\pi}{2}$. i.e., $1 > \frac{sinx}{x} > \frac{2}{\pi}$ Thus we get the desired results.

2.1.14 Show that $\frac{x}{sinx}$ strictly increasing in $(0, \frac{\pi}{2})$. BU 2007

Ans:- Let $f(x) = \frac{x}{sinx}$, then $f'(x) = \frac{sinx - xcosx}{sin^2x} > 0 \ \ \forall x \in (0, \frac{\pi}{2})$,(for details see the just previous example)

2.1.15 Show that $x - \frac{x^2}{6} < sinx < x$ for $x > 0$. BU 2007

Do yourself.

2.1.16 Show that $\frac{x}{1+x} < log_e(1+x)$ for $x > 0$. BU2004

Do yourself.

2.1.17 Verify the conditions of Rolle's theorem for the following functions

(i) $f(x) = cos\frac{1}{x}$ **in [-1,1]**
(ii)(a) $f(x) = 1 - (x-1)^{2/3}$ **in [0,2]:(b)**$f(x) = 1 - (x-1)^2$ **in [0,2]**
(iii)$f(x) = |x - 2|$ **in [1,3]**
(iv) $f(x) = (x-a)^m(x-b)^n$ **in [a,b], for naturals m,n**
(v)$f(x) = x^2 - 5x + 10$ **in [a-1,a+1]**
(vi) $f(x) = |x - a|$ **in [1,3]**
(vii) $f(x) = x$ **for** $-1 < x \leq 1$ **and f(-1)=1 in [-1,1]**
(viii)$f(x) = |x|$ **for** $-1 \leq x \leq 1$

Do yourself.

2.1.18 Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in [0,1].

Ans:- If possible, let α, β be two distinct real roots of the equation $f(x) = x^3 - 3x + k = 0$ in [0,1]. Then $(\alpha, \beta) \subset [0, 1]$. Clearly all the three conditions of Rolle's theorem are satisfied for the function $f(x)$ in (α, β). So by the conclusion of Rolle's theorem the equation $f'(x) = 0$ must have a root in (α, β). But the roots of the equation $f'(x) = 3(x^2 - 1) = 0$ are $x = \pm 1 \notin (\alpha, \beta)$. Hence there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in [0,1].

2.1.19 Prove that if P be any polynomial and P' be its derivative, then between any two consecutive zeros of P' there lies at most one zero of P.

Ans:- If possible, let α, β be two consecutive zeros of P' and there are more than one zeros of P in (α, β). Let c, d be two such zeros of P in (α, β). Then as a polynomial function, clearly P satisfies all the three conditions of Rolle's theorem in $[c, d]$, so there exists at least one $\xi \in (c, d)$ such that ξ is a zero of the polynomial P', which is a contradiction to the fact that α, β are two consecutive zeros of P'. Hence the result.

2.1.20 Show that between any two real roots of $e^x cosx = 1$ there exists at least one root of $e^x sinx - 1 = 0$.

Do youtrself

2.1.21 Prove that if $a_0, a_1, ..., a_n$ are real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + ... + \frac{a_{n-1}}{2} + a_n = 0$ then there exists at least one real number x between 0 and 1 such that $a_0x^n + a_1x^{n-1} + ... + a_n$=0. BU 2008

Ans:- Let us consider the function $f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + ... + \frac{a_{n-1}x^2}{2} + a_nx$.
Clearly the polynomial function $f(x)$ is continuous in [0,1] and is differentiable in (0,1), moreover $f(0) = 0$ and $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + ... + \frac{a_{n-1}}{2} + a_n = 0$. Therefore $f(0) = f(1)$. so all the conditions of Rolle's theorem are satisfied for $f(x)$ on [0,1]. So by the conclusion of the theorem we can say $\exists$ at least one real number x between 0 and 1 such that $a_0x^n + a_1x^{n-1} + ... + a_n$=0.

2.1.22 If $a+b+c=0$ then show that the quadratic equation $3ax^2+2bx+c=0$ has at least one root in (0,1)

Ans: Consider the function $f(x)=ax^2+bx+c$ and apply Rolle's theorem in [0,1].

2.1.23 If $a_0+a_1+....+a_n=0$ where $a_0,a_1,.....,a_n$ are reals, show that the equation $a_0+2a_1x+.....+(n+1)a_nx^n=0$ has at least one root in (0,1)

Ans:- Consider the function $f(x)=a_0x+a_1x^2+....+a_nx^{n+1}$ and apply Rolle's theorem in[0,1].

2.1.24 By considering the function $(x-4)logx$. show that the equation $xlogx=4-x$ is satisfied by at least one value of x lying between 1 and 4.

Do yourself

2.1.25 If P(x) is a polynomial and K is a real number. Prove that between any two real roots of P(x)=0 there is a root of $P'(x)+KP(x)=0$.. BU 2014

Ans:- Consider the function $f(x)=e^{Kx}P(x)\forall x\in R$ and consider α, β as two real zeros of P(x), then use Rolle's theorem.

2.1.26 If f(x) and g(x) are differentiable on (a,b) and continuous on [a,b] and $f(a)=f(b)=0$ then show that there exists a point c $\in(a,b)$ s.t $f'(c)+f(c)g'(c)=0$

Ans:-Consider the function $h(x)=f(x)e^{g(x)}\forall x\in[a,b]$ and then use Rolle's theorem.

2.1.27 If $f(x)=cosx$ find $lim_{h\to0}\theta_h$ where θ_h is given by $f(h)=f(0)+hf'(\theta_hh)$, $0<\theta_h<1$

Ans:- Since $f(h)=f(0)+hf'(\theta_hh)$
$\Rightarrow cosh=1-hsin(\theta_hh)\ \Rightarrow hsin(\theta_hh)=1-cosh=2sin^2(\frac{h}{2})\Rightarrow\theta_h\frac{sin(\theta_hh)}{\theta_hh}=\frac{1}{2}(\frac{sin\frac{h}{2}}{\frac{h}{2}})^2$
Taking the limit as $h\to0$ we have, $lim_{h\to0}\theta_h=\frac{1}{2}$

3 Descriptive type questions

3.0.28 State and prove Taylor's theorem with a remainder after n terms. BU2001,2004,2007,2009,2013

3.0.29 State and prove Lagrange's mean value theorem . BU2006

3.0.30 State and prove Cauchy's mean value theorem. Hence deduce Lagrange's mean value theorem. BU2006,2008,2012,2014

3.0.31 Obtain Taylor series expansion of $log_e(1+x)$ indicating the range of convergence. BU2006

3.0.32 Apply Maclaurin's theorem to obtain power series expansion of $log_e(1+x)$ for $|x|<1$. BU2001, 2007, 2008

Ans:-Let $f(x) = log_e(1+x)$, then $f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$
Which exists for every value of n and for $x > -1$
Now $f^n(0) = (-1)^{n-1}(n-1)!$, Now the following two cases may arise
Case-1: Let $0 \leq x \leq 1$. If R_n denotes the Lagrange's form of remainder in the Maclaurin's power series expansion of $log_e(1+x)$, the we have
$R_n = \frac{x^n}{n!}f^n(\theta x) = \frac{(-1)^{n-1}}{n}(\frac{x}{1+\theta x})^n$.
Now $(\frac{x}{1+\theta x})^n \to 0$ as $n \to \infty$, since $\frac{x}{1+\theta x}$ is positive and less than 1 for $0 \leq x \leq 1$.
Also $\frac{1}{n} \to 0$ as $n \to \infty$.
So $R_n \to 0$ as $n \to \infty$.
Case-II: Let $-1 < x < 0$, in this case $\frac{x}{1+\theta x}$ may not be numerically less than unity and hence $(\frac{x}{1+\theta x})^n$ may not be tend to 0 as $n \to \infty$.
Thus Lagrange's form of remainder in the Maclaurin's power series expansion do not give us definite conclusion. Now using the Cauchy's form of remainder, we have
$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x) = (-1)^{n-1}\frac{x^n}{1+\theta x}(\frac{1-\theta}{1+\theta x})^{n-1}$
Now $\frac{1-\theta}{1+\theta x}$is positive and less than 1, hence $(\frac{1-\theta}{1+\theta x})^{n-1} \to 0$ as $n \to \infty$.
Also $x^n \to 0$ as $n \to \infty$, since in $-1 < x < 0$; $\frac{1}{1+\theta x}$ is bounded and hence, $R_n \to 0$ as $n \to \infty$.
Thus the Maclaurin's power series expansion of $log_e(1+x)$ is given by
$log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} -$ ...., where $-1 < x \leq 1$.

3.0.33 Establish the inequality $\frac{x^2}{2} < x - log_e(1+x) < \frac{x^2}{2(1+x)}$ when $-1 < x < 0$. BU2001

Do yourself

3.0.34 State and prove Maclaurin's theorem with Lagrange's form of remainder. BU2002, 2005

3.0.35 Obtain Maclaurin's infinite series expansion of $(1+x)^m$ where m is any real number other than positive integer and $|x| < 1$. BU2002, 2006, 2008

See any book

3.0.36 Assuming $f''(x)$ to be continuous on [a,b], show that $f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$, where c and ξ both lie in [a,b]. BU2007

3.0.37 If for a function f defined in a nbd of a, f' is continuous at a and $f''(a) \neq 0$ then prove that $lim_{h \to 0+}\theta = \frac{1}{2}$, where θ is given by $f(a+h) = f(a) + hf'(a+\theta h), \quad 0 < \theta < 1$ BU 2012, 2014

3.0.38 Use MVT to prove that $\frac{x-1}{x} < log_e(x) < x-1$ when $x > 1$. BU2013

Do youself

3.0.39 State and prove Rolle's theorem . BU 2011

3.0.40 Expand the function $h(x) = sin^2 x$ in a nbd of x=0 to three terms plus remainder in Lagrange's form. BU 2012

Do yourself

Dr. M K Ghorui Wednesday, July 23, 2014