1 System of Particles

Since whenever we discussed about the motion of an object, we consider the object as point like, generally known as a particle but in practical an object is a collections or system of particles. The system of particles may be discrete or continuous according as the particles can be considered as separated from each other or not. A discrete system having a very large but finite number of particles can be considered as a continuous system. Conversely a continuous system can be considered as a discrete system consisting of a large but finite number of particles. We know in a system of particles, the particles move under the action of two types of forces, namely
(i) **Internal forces:-** Internal forces aries as a result of the inter-action of the particles with other particles of the system. The totality of these forces may be considered to be pairs of equal and opposite forces having same line of action.
(ii)**External forces:-**The forces other than the internal forces are called the external forces. Thus external forces are exerted by the external agents of the system whereas internal forces are the bounding forces of interaction among the constituent particles of the system. For example gravitational, magnetic, frictional forces are the external forces.
Forces applied to systems of particles will changes the distances between individual particles. Such systems are often called deformable or elastic bodies. If the deformations be so small that it is to be considered as non-existent then the system is called a rigid body. Thus a mathematical model in which the distance between any two specified particles of a system remains the same regardless of the applied forces, is called a rigid body.
The number of independent coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.
Examples:-(1). A particle moving freely in space requires three co-ordinates, e.g. (x,y,z) to specify its position. Thus the number of degrees of freedom is three.
(ii)A system consisting of of N particles moving freely in space requires 3N co-ordinates to specify its position.Thus the number of degrees of freedom is 3N.
(iii) If three non -collinear points of a rigid body are fixed in space,then the rigid body is also fixed in space.Let these points have c o-ordinates $(x_i, y_i, z_i), i = 1, 2, 3$ respectively.Thus there are 9 co-ordinates.Since the body is rigid we must have the relations $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = \text{constant}$,where $i \neq j; i, j = 1, 2, 3$
Hence 3 co-ordinates can be expressed in terms of the remaining 6.Thus 6 independents co-ordinates are needed to describe the motion ,i,e,there are 6 degrees of freedom.
(iv)Let a particle is moving on a given space curve ,whose parametric equation is

given by $x = x(s), y = y(s), z = z(s)$where s is a parameter.Thus the position of a particle on the curve is determined by specifying one co-ordinates and hence there is one degree of freedom.
(v)If four particles moving freely in a plane then each particle requires two co-ordinates to specify its position in the plane.Thus $4 \times 2 = 8$ co-ordinates are needed to specify the positions of all four particles i,e; the system has 8 degrees of freedom.
(vi)If four particles moving freely in space then each particle requires three co - ordinates to specify its position in space.The system has $4 \times 3 = 12$ degrees of freedom.
(vii)If two particles are connected by a rigid rod moving freely in a plane then there are 3 degrees of freedom of the system ,because if $(x_i, y_i,), i = 1, 2,$ be the co-ordinates of the two particles i.e, a total of 4 co-ordinates are needed to specify the system but since the distance between these two points is a constant.So we have $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$(constant).So one of the co-ordinates can be expressed in terms of the others.Thus there are $4 - 1 = 3$ degrees of freedom.
(viii)If a rigid body has one point fixed but can move in space about this point then the motion is completely specified if we know the co-ordinates of the two points say $(x_i, y_i, z_i), i = 1, 2,$ where the fixed point is taken at the origin of co-ordinate system. But since the body is rigid ,we must have
$(x_1)^2 + (y_1)^2 + (z_1)^2 =$ constant,$(x_2)^2 + (y_2)^2 + (z_2)^2 =$ constant and
$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 =$ constant.
from which 3 co-ordinates can be found in terms of the remaining 3.Thus there are 3 degrees of freedom.

2 Centre of mass

Let r_i be the position of the i-th particle of a system of N particles m_i be the mass of the i-th particle .The centre of mass or centroid of the system of particles is defined as that point having position vector
$\vec{r} = \frac{m_1\vec{r_1}+m_2\vec{r_2}+.....+m_N\vec{r_N}}{m_1+m_2+.....+m_N}$
$= \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r_i}$
Where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system.

3 Momentum or linear momentum of a system of particles

If $\vec{v_i} = \frac{d\vec{r_i}}{dt} = \dot{\vec{r_i}}$ is the velocity of the i^{th} particle of mass m_i of a system of particles then the total linear momentum of the system is defined as $\vec{p} = \sum_{i=1}^{N} m_i\vec{v_i} = \sum_{i=1}^{N} m_i\dot{\vec{r_i}}$. Since if $\vec{r}$ be the position vector of the center of mass then $M\vec{r} = \sum_{i=1}^{N} m_i\vec{r_i} \Rightarrow M\dot{\vec{r}} = \sum_{i=1}^{N} m_i\dot{\vec{r_i}} = \vec{p} \Rightarrow \vec{p} = M\dot{\vec{r}} = M\vec{v}$ where $\vec{v} = \dot{\vec{r}}$ is the velocity of the center of mass of the system. Thus the total momentum of a system of particles can be found by multiplying the total mass M of the system by the velocity $\vec{v}$ of the center of mass.

4 Motion of the center of mass

Theorem:- The center of mass of a system of particles moves as if the total mass and resultant external force were applied at this point.

Proof:-Let $\vec{F_i}$ be the resultant external force acting on the i^{th} particle of mass m_i of a system of N particles and $\vec{f_{ij}}$ be the internal force on the i^{th} particle due to the j^{th} particle. We shall assume that $\vec{f_{ii}} = \vec{0}$, i.e. the particle i does not exert any force on itself.

By Newton's second law of motion the total force on the i^{th} particle is

$\vec{F_i} + \sum_{j=1,i\neq j}^{N} \vec{f_{ij}} = \frac{d\vec{p_i}}{dt} = \frac{d^2}{dt^2}(m_i\vec{r_i})$

where the second term on the left hand side represents the resultant internal force on the i^{th} particle due to the other (N-1) particles of the system. $\vec{r_i}$ be the position vector, m_i be the mass and $\vec{p_i}$ is the linear momentum of the i^{th} particle of the system. Summing over i in the above equation we have

$\sum_{i=1}^{N} \vec{F_i} + \sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N} \vec{f_{ij}} = \frac{d^2}{dt^2}(\sum_{i=1}^{N} m_i\vec{r_i})$

Now according to Newton's third law of action and reaction $\vec{f_{ij}} = -\vec{f_{ji}}$. So that the double summation on the L.H.S of the above equation is zero and then the above equation can be written as

$\vec{F} = M\frac{d^2\vec{r}}{dt^2}$

where $\vec{F} = \sum_{i=1}^{N} \vec{F_i}$ and $\vec{r} = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r_i}$. Since $\vec{F}$ is the total external force on all the particles applied at the center of mass having position vector $\vec{r}$. Hence the required result is proved.

Theorem:- If the total momentum of a system is constant, i.e. is conserved then the center of mass is either at rest or in motion with constant velocity.

Proof:- We know that if $\vec{r_i}$ be the position vector of the i^{th} particle of mass m_i moving with velocity $\vec{v_i}$, then the total momentum of the system of N particles is given by

$\vec{p} = \sum_{i=1}^{N} m_i \vec{v_i} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}} = \frac{d}{dt}(\sum_{i=1}^{N} m_i \vec{r_i}) = M \frac{d\vec{r}}{dt}$
where $\vec{r}$ is the position vector of the center of mass of the system of N particles. Now if $\vec{p}$ is constant, then $\frac{d\vec{r}}{dt}$ is also constant. Thus the center of mass is either at rest or in motion with constant velocity.

5 Conservation of linear momentum

Statement:-If the resultant external force acting on a system of particles is zero, then the total linear momentum remains constant, i.e. is conserved.
Proof:- We know that if $\vec{r_i}$ be the position vector of the i^{th} particle of mass m_i moving with velocity $\vec{v_i}$, then the total momentum of the system of N particles is given by
$\vec{p} = \sum_{i=1}^{N} m_i \vec{v_i} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}} = \frac{d}{dt}(\sum_{i=1}^{N} m_i \vec{r_i}) = M \frac{d\vec{r}}{dt} = M\dot{\vec{r}}$
Let the internal forces between any two particles of the system obey Newton's third law. Then if $\vec{F}$ is the resultant external force acting on the system, we have
$\vec{F} = \frac{d\vec{p}}{dt} M \frac{d^2\vec{r}}{dt^2} = M\ddot{\vec{r}}$
Now if $\vec{F} = 0$, then we have, $\ddot{\vec{r}} = \frac{d^2}{dt^2}(\sum_{i=1}^{N} m_i \vec{r_i}) = 0 \Rightarrow \frac{d}{dt}(\sum_{i=1}^{N} m_i \vec{v_i}) = 0$. Integrating we have $\vec{p} = \sum_{i=1}^{N} m_i \vec{v_i} =$ constant. Hence the result.

6 Angular Momentum or Moment of momentum of a system of particles

If $\vec{r_i}$ be the position vector of the i^{th} particle of mass m_i moving with velocity $\vec{v_i}$, of the system consisting N particles, then the quantity $\vec{\Omega} = \sum_{i=1}^{N}(\vec{r_i} \times m_i \vec{v_i})$ is called the total angular momentum or angular momentum of the system of particles about the origin O.

7 External torque acting on a system of particles

If $\vec{F_i}$ is the external force acting on the i^{th} particle having position vector $\vec{r_i}$ of a system of N particles, then $\vec{r_i} \times \vec{F_i}$ is called the moment of the force $\vec{F_i}$ or torque about 0. The sum $\vec{\triangle} = \sum_{i=1}^{N}(\vec{r_i} \times \vec{F_i})$ is called the total external torque about the origin.
Theorem:- The total external torque on a system of particles is equal to the time

rate of change of the angular momentum of the system, provided the internal forces between the particles are central forces.
Proof:- Let $\vec{F_i}$ be the resultant external force acting on the i^{th} particle of mass m_i of a system of N particles and $\vec{f_{ij}}$ be the internal force on the i^{th} particle due to the j^{th} particle. We shall assume that $\vec{f_{ii}} = \vec{0}$, i.e. the particle i does not exert any force on itself.
By Newton's second law of motion the total force on the i^{th} particle is
$\vec{F_i} + \sum_{j=1,i\neq j}^{N} \vec{f_{ij}} = \frac{d\vec{p_i}}{dt} = \frac{d^2}{dt^2}(m_i\vec{r_i}) = \frac{d}{dt}(m_i\vec{v_i})$
where the second term on the left hand side represents the resultant internal force on the i^{th} particle due to the other (N-1) particles of the system. $\vec{r_i}$ be the position vector, m_i be the mass and $\vec{p_i}$ is the linear momentum of the i^{th} particle of the system.
From the above equation we can write
$\vec{r_i} \times \vec{F_i} + \sum_{j=1,i\neq j}^{N} \vec{r_i} \times \vec{f_{ij}} = \vec{r_i} \times \frac{d\vec{p_i}}{dt} = \vec{r_i} \times \frac{d^2}{dt^2}(m_i\vec{r_i}) = \vec{r_i} \times \frac{d}{dt}(m_i\vec{v_i}) = \frac{d}{dt}(m_i\vec{r_i} \times \vec{v_i})$
since $\vec{v_i} \times \vec{v_i} = 0$. Summing over i we have
$\sum_{i=1}^{N} \vec{r_i} \times \vec{F_i} + \sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N} \vec{r_i} \times \vec{f_{ij}} = \frac{d}{dt}(\sum_{i=1}^{N} m_i\vec{r_i} \times \vec{v_i})$
Now the double sum in the L.H.S of the above equation is composed of terms like $\vec{r_i} \times \vec{f_{ij}} + \vec{r_j} \times \vec{f_{ji}}$, which becomes $(\vec{r_i} - \vec{r_j}) \times \vec{f_{ij}}$ on writing $\vec{f_{ij}} = -\vec{f_{ji}}$ according to the Newton's third law of motion.Thus,
$\vec{r_i} \times \vec{f_{ij}} + \vec{r_j} \times \vec{f_{ji}} = (\vec{r_i} - \vec{r_j}) \times (\vec{f_{ij}})$
Now if the forces are central then $\vec{f_{ij}}$ and $(\vec{r_i} - \vec{r_j})$ have the same direction. It follows that the above expression is zero and also the double sum in the above equation is zero. Thus the above equation becomes
$\sum_{i=1}^{N} \vec{r_i} \times \vec{F_i} = \frac{d}{dt}(\sum_{i=1}^{N} m_i(\vec{r_i} \times \vec{v_i}))$
$\Rightarrow \vec{\triangle} = \frac{d\vec{\Omega}}{dt}$
where $\vec{\triangle} = \sum_{i=1}^{N} \vec{r_i} \times \vec{F_i}$ is the total external torque acting on the system about the origin and $\vec{\Omega} = \sum_{i=1}^{N} m_i(\vec{r_i} \times \vec{v_i})$ is the total angular momentum of the system about the origin. Hence the result.

8 Conservation of angular momentum:-

STATEMENT:-If the resultant external torque acting on a system of particles is zero then the total angular momentum remains constant i,e. is conserved.
Proof:-Let $\vec{F_i}$ be the resultant external force acting on the i^{th} particle of mass m_i of a system of N particles and $\vec{f_{ij}}$ be the internal force on the i^{th} particle due to the j^{th} particle. We shall assume that $\vec{f_{ii}} = \vec{0}$, i.e. the particle i does not exert any force on

itself.
By Newton's second law of motion the total force on the i^{th} particle is
$\vec{F_i} + \sum_{j=1,i\neq j}^{N} \vec{f_{ij}} = \frac{d\vec{p_i}}{dt} = \frac{d^2}{dt^2}(m_i\vec{r_i}) = \frac{d}{dt}(m_i\vec{v_i})$
where the second term on the left hand side represents the resultant internal force on the i^{th} particle due to the other (N-1) particles of the system. $\vec{r_i}$ be the position vector, m_i be the mass and $\vec{p_i}$ is the linear momentum of the i^{th} particle of the system.
From the above equation we can write
$\vec{r_i} \times \vec{F_i} + \sum_{j=1,i\neq j}^{N} \vec{r_i} \times \vec{f_{ij}} = \vec{r_i} \times \frac{d\vec{p_i}}{dt} = \vec{r_i} \times \frac{d^2}{dt^2}(m_i\vec{r_i}) = \vec{r_i} \times \frac{d}{dt}(m_i\vec{v_i}) = \frac{d}{dt}(m_i\vec{r_i} \times \vec{v_i})$
since $\vec{v_i} \times \vec{v_i} = 0$. Summing over i we have
$\sum_{i=1}^{N} \vec{r_i} \times \vec{F_i} + \sum_{i=1}^{N}\sum_{j=1,i\neq j}^{N} \vec{r_i} \times \vec{f_{ij}} = \frac{d}{dt}(\sum_{i=1}^{N} m_i\vec{r_i} \times \vec{v_i})$
Now the double sum in the L.H.S of the above equation is composed of terms like $\vec{r_i} \times \vec{f_{ij}} + \vec{r_j} \times \vec{f_{ji}}$, which becomes $(\vec{r_i} - \vec{r_j}) \times \vec{f_{ij}}$ on writing $\vec{f_{ij}} = -\vec{f_{ji}}$ according to the Newton's third law of motion.Thus,
$\vec{r_i} \times \vec{f_{ij}} + \vec{r_j} \times \vec{f_{ji}} = (\vec{r_i} - \vec{r_j}) \times (\vec{f_{ij}})$
Now if the forces are central then $\vec{f_{ij}}$ and $(\vec{r_i} - \vec{r_j})$ have the same direction. It follows that the above expression is zero and also the double sum in the above equation is zero. Thus the above equation becomes
$\sum_{i=1}^{N} \vec{r_i} \times \vec{F_i} = \frac{d}{dt}(\sum_{i=1}^{N} m_i(\vec{r_i} \times \vec{v_i}))$
$\Rightarrow \vec{\triangle} = \frac{d\vec{\Omega}}{dt}$
where $\vec{\triangle} = \sum_{i=1}^{N} \vec{r_i} \times \vec{F_i}$ is the total external torque acting on the system about the origin and $\vec{\Omega} = \sum_{i=1}^{N} m_i(\vec{r_i} \times \vec{v_i})$ is the total angular momentum of the system about the origin. Now if $\vec{\triangle} = \vec{0}$ then we have $\vec{\Omega} = \sum_{i=1}^{N} m_i(\vec{r_i} \times \vec{v_i})$ = constant.
Hence the result.

9 Kinetic energy of a system of particles:-

If $\vec{r_i}$ be the position vector of the i^{th} particle of mass m_i of a system of N particles then total kinetic energy of the system is defined as
$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r_i}^2$
where v_i is the velocity of the i^{th} particle.

10 Work:-

If $\vec{\boldsymbol{F}}_i$ is the force (external and internal) acting on the i^{th} particle having position vector $\vec{r_i}$ of a system of N particles , then the total work done in moving the system of particles from a system s_1 to s_2 is
$W_{12} = \sum_{i=1}^{N} \int_{s_1}^{s_2} \vec{\boldsymbol{F}}_i.d\vec{r_i}$ where $\vec{\boldsymbol{F}}_i = \vec{F_i} + \sum_{j=1,j\neq i}^{N} \vec{f_{ij}}$

11 Conservative forces and Potential energy:-

Forces acting on a body are said to be conservative if the work done by the forces in causing displacement of the body depends only on the initial and final positions of the body and not on the path traversed from the initial to final position. Hence the work done by the conservative force $\vec{F}$ along an arbitrary closed path of its point of application is zero. Thus $\oint \vec{F}.d\vec{r} = 0$
By Stokes's theorem the above equation may be written as $curl\vec{F} = \vec{\nabla} \times \vec{F} = 0$
Since the curl of a gradient always vanishes, so $\vec{F}$ must be gradient of a scalar (potential energy). Hence $\vec{F} = -\vec{\nabla}V = -grad\ V$, V is called the potential energy. If the friction or other dissipation forces are present, the forces cannot be said to be conservative **Theorem:** The total work done in moving a system of particles from one state where the kinetic energy is T_1 to another state where the kinetic energy is T_2 is $W_{12} = T_2 - T_1$
Proof: With the usual notations as earlier the equation of motion of the i-th particle in a system of N particles is given by
$\vec{\boldsymbol{F}}_i = \vec{F_i} + \sum_{j=1,j\neq i}^{N} \vec{f_{ij}} = \frac{d}{dt}(m_i\dot{\vec{r_i}})$———(1)
Taking the dot product of both sides with $\dot{r}_i$ we have
$\vec{\boldsymbol{F}}_i.\dot{\vec{r_i}} = \vec{F_i}.\dot{\vec{r_i}} + \sum_{j=1,j\neq i}^{N} \vec{f_{ij}}.\dot{\vec{r_i}} = \frac{1}{2}\frac{d}{dt}(m_i\dot{r_i}^2) = \frac{1}{2}\frac{d}{dt}(m_iv_i^2)$——-(2)
Summing over i in eqn (2)we have
$\sum_{i=1}^{N} \vec{\boldsymbol{F}}_i.\dot{\vec{r_i}} = \sum_{i=1}^{N} \vec{F_i}.\dot{\vec{r_i}} + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \vec{f_{ij}}.\dot{\vec{r_i}} = \frac{1}{2}\frac{d}{dt}(\sum_{i=1}^{N} m_iv_i^2)$——-(3)
Integrating both sides with respect to t from t_1 to t_2 we have,
$W_{12} = \sum_{i=1}^{N}\int_{t_1}^{t_2} \vec{\boldsymbol{F}}_i.\dot{\vec{r_i}}dt = \sum_{i=1}^{N}\int_{t_1}^{t_2} \vec{F_i}.\dot{\vec{r_i}}dt + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\int_{t_1}^{t_2} \vec{f_{ij}}.\dot{\vec{r_i}}dt = \frac{1}{2}\sum_{i=1}^{N}\int_{t_1}^{t_2} \frac{d}{dt}(m_iv_i^2)dt$
Let S_1 represents the state-1 at time $t = t_1$ and S_2 represents the state-2 at time $t = t_2$ of the system. Thus from the above equation we have
$W_{12} = \sum_{i=1}^{N}\int_{t_1}^{t_2} \vec{\boldsymbol{F}}_i.d\vec{r_i} = \sum_{i=1}^{N}\int_{t_1}^{t_2} \vec{F_i}.d\vec{r_i} + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\int_{t_1}^{t_2} \vec{f_{ij}}.d\vec{r_i} = T_2 - T_1$
where T_1 and T_2 are the total K.E at t_1 and t_2 respectively. Since $W_{12} = \sum_{i=1}^{N}\int_{t_1}^{t_2} \vec{\boldsymbol{F}}_i.d\vec{r_i}$ is the total work done (by external and internal forces) in moving the system from one state to another. Hence the result.

THEOREM: If the internal and the external forces are conservative then show that the total work done in moving a system from initial configuration 1 to final configuration 2 is given by $W_{12} = V_1 - V_2$
where V_1, V_2 are the total potential energies in configuration-1 and configuration-2 respectively.
proof:-We know that the total work done in moving a system by all the forces - external as well as internal from initial configuration-1 to final configuration-2 is equal to the sum of the work done in moving all the particles from configuration-1 to configuration-2.Thus
$W_{12} = \sum_i \int_1^2 \vec{F_i}.d\vec{r_i} = \sum_i \int_1^2 \vec{F_i}^{ext}.d\vec{r_i} + \sum'_{ij} \int_1^2 \vec{F_{ij}}^{int}.d\vec{r_i}$——(1)
The prime over the summation symbolizes that the term with i=j in the sum is excluded.
If the internal and the external forces are conservative, then they can be expressed in terms of corresponding potential energies.Thus total force $\vec{F_i}$ on the i-th particle can be written as
$\vec{F_i} = \vec{F_i}^{ext} + \sum_j \vec{F_{ij}}^{int} = -\vec{\nabla}_i V_i$——-(2)
where the potential energy
$V_i = V_i^{ext} + V_i^{int}$——(3)
is the sum of potential energy functions of the external and internal forces.In equation (2),symbol $\vec{\nabla}_i$ is
$\vec{\nabla}_i = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$ ———-(4)
and it represents the gradient operator performing differentiation with respect to x_i,components of position vector $\vec{r_i}$ of the i-th particle . The operator can be written separately as
$\vec{F_i}^{ext} = -\vec{\nabla}_i V_i^{ext}$
and $\vec{F_i^{int}} = -\vec{\nabla}_{ij} V_{ij}^{int}$———(5)
Quantity V_{ij}^{int} is the potential energy arising due to internal forces $\vec{F_{ij}}^{int}$ and $\vec{\nabla}_{ij} = \sum_i \hat{e}_i \frac{\partial}{\partial(x_i - x_j)}$.From this ,it will be clear that $\vec{\nabla}_{ij} = -\vec{\nabla}_{ji}$.
Now the potential energy of the i^{th}particle arising due to internal forces is given by
$V_i^{int} = \sum_j V_{ij}^{int}$
Hence ,the total potential energy due to internal forces is
$V^{int} = \sum_i V_i^{int} = \sum_{ij, i<j} V_{ij}^{int}$——-(6)
We have to take $V_{ii}^{int} = 0$ to have $F_{ii}^{int} = 0$.Condition $i < j$ is necessary because otherwise each term will be taken twice in summing over i and j. Potential energy V_{ij}^{int} depends upon the relative positions of the two particles i.e,$V_{ij}^{int} \equiv V_{ij}^{int}(r_{ij})$.
Then by Newton's third law, we have
$\vec{F_{ij}^{int}} = -\vec{\nabla}_{ij} V_{ij}^{int} = -\vec{F_{ji}^{int}} = \vec{\nabla}_{ji} V_{ji}^{int} = -\vec{\nabla}_{ij} V_{ji}^{int}$———(7)

From (7),we find that $V_{ij}^{int} = V_{ji}^{int}$ and hence $V^{int} = \frac{1}{2}\sum_{ij} V_{ij}^{int}$———(8)
Factor $\frac{1}{2}$ in (8)is due to the fact that each term is being taken twice while summing over i and j.The same has been incorporated in equation (6)by writing $i < j$ .This condition avoids the duplication of terms. The work done by the external forces is given by

$$\sum_i \int_1^2 \vec{F}_i^{ext}.d\vec{r_i} = -\sum_i \int_1^2 \vec{\nabla}_i V_i^{ext}.d\vec{r_i} = -\sum_i \int_1^2 dV_i^{ext} = -[\sum_i V_i^{ext}]_1^2 = V_1^{ext} - V_2^{ext} \quad (9)$$

where V_1^{ext} and V_2^{ext} represent the potential energies of the system arising due to external forces acting on the system in configuration-1 and configuration-2 respectively.
The work done by the internal forces is given by $\sum'_{ij} \int_1^2 \vec{F_{ij}}^{int}.d\vec{r_i}$
But $\sum'_{ij} \vec{F_{ij}}^{int}.d\vec{r_i} = \sum'_{ji} \vec{F_{ji}}^{int}.d\vec{r_j} = -\sum'_{ij} \vec{F_{ij}}^{int}.d\vec{r_j}$
Hence $\sum'_{ij} \vec{F_{ij}}^{int}.d\vec{r_i} = \frac{1}{2}\sum'_{ij} \vec{F_{ij}}^{int}.(d\vec{r_i} - d\vec{r_j}) = \frac{1}{2}\sum'_{ij} \vec{F_{ij}}^{int}.d\vec{r_{ij}}$
where $d\vec{r_{ij}} = d\vec{r_i} - d\vec{r_j}$
Substituting this value we get

$$\sum'_{ij} \vec{F_{ij}}^{int}.d\vec{r_i} = \frac{1}{2}\sum'_{ij} \int_1^2 \vec{F_{ij}}^{int}.d\vec{r_{ij}} = -\frac{1}{2}\sum'_{ij} \int_1^2 \vec{\nabla}_{ij} V_{ij}^{int}.d\vec{r_{ij}} = -\frac{1}{2}\sum'_{ij} \int_1^2 dV_{ij}^{int}$$
$$= -[\frac{1}{2}\sum'_{ij} V_{ij}^{int}]_1^2 = [-V^{int}]_1^2 = V_1^{int} - V_2^{int} \text{ (by (8))}.$$

Now the total energy of the system is given by
$V = V^{ext} + V^{int} = \sum_i V_i^{ext} + \sum_{ij,i<j} V_{ij}^{int}$
Thus in terms of the total potential energy of the system, the work done is give by
$W_{12} = [-V]_1^2 = V_1 - V_2$
Note :-combining the above to theorems we get $T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2$.Which states that the total energy of the system is conserved. Thus when all the forces, external and internal, are conservative, the total energy of the system $i.e.T + V$ is conserved. This is the principle of conservation of energy for systems of particles.

12 Motion relative to the center of mass:-

Theorem:-The total linear momentum of a system of particles about the center of mass is zero.
Proof:- Let $\vec{r_i}$ be the position vector of the i^{th} particle relative to the origin 'O' and $\vec{r}$ be the position vector of the center of mass relative to the origin 'O'. Let $\vec{r_i}'$ and $\vec{v_i}'$ be the position vector and velocity of the i^{th} particle to the center of mass C. Then from definition of the center of mass we have
$\vec{r} = \frac{1}{M}\sum_i m_i\vec{r_i}$ (1) where $M = \sum_i m_i$
Now from the triangle law of vector we have $\vec{r_i} = \vec{r_i}' + \vec{r}$ (2)
Using (2)in (1)we have $\vec{r} = \frac{1}{M}\sum_i m_i(\vec{r_i}' + \vec{r}) = \frac{1}{M}\sum_i m_i\vec{r_i}' + \vec{r}$

$\sum_i m_i\vec{r_i}' = 0$
Differentiating both sides w.r.to t we have $\sum_i m_i\dot{\vec{r_i}}' = \sum_i m_i\vec{v_i}' = 0$. Hence the result.
Theorem:- The total angular momentum of a system of particles about any point o equals the angular momentum of the total mass assumed to be located at the center of mass plus the angular momentum about the center of mass.In symbols $\vec{\Omega} = \vec{r} \times M\vec{v} + \sum_{i=1}^{N} m_i(\vec{r_i})' \times \vec{v_i}'$
Proof:- Let $\vec{r_i}$ be the position vector of the i^{th} particle and $\vec{r}$ be the p.v of the center of mass of a system of N particles relative to 'O'. Let $\vec{r_i}'$ be the p.v of the i^{th} particle relative to the center of mass C.
Now from the triangle law of vector we have $\vec{r_i} = \vec{r_i}' + \vec{r}$ (1)
Differentiating both sides w.r.to t we have, $\vec{v_i} = \dot{\vec{r_i}} = \dot{\vec{r_i}}' + \dot{\vec{r}} = \vec{v_i}' + \vec{v}$ (2)
where $\vec{v}$ is the velocity of c.m relative to o,$\vec{v_i}$ is the velocity of the i^{th} particle relative to o and $\vec{v_i}'$ is the velocity of the i^{th} particle relative to C
The total angular momentum of the system about o is
$\vec{\Omega} = \sum_i m_i(\vec{r_i} \times \vec{v_i}) = \sum_i m_i\{(\vec{r_i}' + \vec{r}) \times (\vec{v_i}' + \vec{v})\}$
$= \sum_i m_i(\vec{r_i}' \times \vec{v_i}') + \sum_i m_i(\vec{r_i}' \times \vec{v}) + \sum_i m_i(\vec{r} \times \vec{v_i}') + \sum_i m_i(\vec{r} \times \vec{v})$
$= \sum_i m_i\{(\vec{r_i}' + \vec{r}) \times (\vec{v_i}' + M(\vec{r} \times \vec{v})$
Since by the just previous theorem we have $\sum_i m_i\vec{r_i}' = 0$ and $\sum_i m_i\vec{v_i}' = 0$, (proved)
Theorem:-The total K.E. of a system of particles about any point o equals the K.E. of translation of the center of mass (assuming the total mass located there)plus the K.E. of the motion about the center of mass. In symbols $T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2$
Proof:-Let $\vec{r_i}$ be the position vector of the i^{th} particle and $\vec{r}$ be the p.v of the center of mass of a system of N particles relative to 'O'. Let $\vec{r_i}'$ be the p.v of the i^{th} particle relative to the center of mass C.
Now from the triangle law of vector we have $\vec{r_i} = \vec{r_i}' + \vec{r}$ (1)
Differentiating both sides w.r.to t we have, $\vec{v_i} = \dot{\vec{r_i}} = \dot{\vec{r_i}}' + \dot{\vec{r}} = \vec{v_i}' + \vec{v}$ (2)
where $\vec{v}$ is the velocity of c.m relative to o,$\vec{v_i}$ is the velocity of the i^{th} particle relative to o and $\vec{v_i}'$ is the velocity of the i^{th} particle relative to C
Now the K.E. of the system relative to o is
$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r_i}}.\dot{\vec{r_i}}) = \frac{1}{2}\sum_{i=1}^{N} m_i\{(\vec{v_i}' + \vec{v}).(\vec{v_i}' + \vec{v})\}$
$= \frac{1}{2}\sum_{i=1}^{N} m_i(\vec{v}.\vec{v}) + \sum_{i=1}^{N} m_i(\vec{v}.\vec{v}') + \frac{1}{2}\sum_{i=1}^{N} m_i(\vec{v}'.\vec{v}')$
$= \frac{1}{2}Mv^2 + \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2$ Since $\sum_i m_i\vec{v_i}' = 0$. Hence the result.

13 Constraints:-

Constraints are the limitations or restrictions on motion of a particle or system of particles and the forces responsible for the restrictions are called the forces of constraints. Condition imposed on the system by the constraints can,in most cases, be written done mathematically as a relation satisfied by the co-ordinates of the particle at any time. This is the way in which the constraints reduce the number of co-ordinates needed to specify the configuration of a system.

13.1 Holonomic and Non-holonomic constraints:-

If the conditions of constraints can be expressed as equations connecting the co-ordinates of the particles (and the time)as $f(r_1, r_2, ...r_n, t) = 0$, then constraints are called holonomic.The constraints not expressible in this way are called Non-holonomic constraints.
For example, in case of a rigid body, the constraints can be expressed as $(r_i - r_j)^2 = c_{ij}^2$, so these are holonomic constraints. A particle constrained to move along a curve or on a surface is another example of holonomic constraint.
On the other hand, the motion of the particle placed on the surface of sphere under the action of gravitational force is bound by Non-holonomic constraints, for it can be expressed as an inequality, $r^2 - a^2 \geq 0$.
The constraints involved in the motion of the molecules in a gas container are Non-holonomic.

13.2 Scleronomic and Rheonomic constraints:-

The constraints, which are independent of time are known as Scleronomic constraints. Rheonomic constraints are those, which depends on time.
A pendulum with a fixed support is Scleronomic, whereas the pendulum for which the point of support is given an assigned motion is Rheonomic.

14 Rigid bodies:-

A system of particles in which the distance between any two particles does not change regardless of the forces acting is called a rigid body. Thus a rigid body is a special case of system of particles. So all the theorems on system of particles are also valid for rigid bodies.

15 Assumptions in a Rigid body:-

In a rigid body it is assumed that
(i) the action between any two particles of a rigid body acts along the straight line joining them and
(ii) the action and reaction between any two particles of a rigid body are equal and opposite.

16 Virtual displacement:-

Consider two possible configurations of a system of particles at a particular instant which are consistent with the forces and constraints. To go from one configuration to the other, we need only give the i^{th} particle a displacement δr_i from the old to the new position. We call δr_i a virtual displacement to distinguish it from a true displacement. Virtual displacement occurs in a time interval where forces and constraints could be changing.

17 Principle of Virtual Work:-

We know that a system of particles is in equilibrium if the resultant force acting on each particle is zero, i.e.$\vec{F_i} = 0$. It thus follows that $\vec{F_i}.d\vec{r_i} = 0$, where $\vec{F_i}.d\vec{r_i}$ is called the virtual work. By adding all these we then have $\sum_{i=1}^{N} \vec{F_i}.\delta\vec{r_i} = 0$. Now if the constraints are present, the we can write $\vec{F_i} = \vec{F_i}^{(a)} + \vec{F_i}^{(c)}$, where $\vec{F_i}^{(a)}$ and $\vec{F_i}^{(c)}$ are respectively the actual force and constraint force on the i^{th} particle. Now assuming the virtual work of the constraint force is zero (which is in case of rigid bodies and for the motion on frictionless curves and surfaces), what we have is known as Principle of Virtual Work. Thus the principle of virtual work states that, a system of particles is in equilibrium iff the total virtual work of the actual forces is zero.
i.e. if $\sum_{i=1}^{N} \vec{F_i}^{(a)}.\delta\vec{r_i} = 0$.