Real AnalysisI, Sem-1, (Hons)Group-A(30 marks)

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Chapter-2:-

Real Sequences:-

1.**Real Sequence:-**
A function whose domain is the set of natural numbers **N** and whose range is the set of real numbers **R** is called a real sequence or simply a sequence. A sequence is generally represented by $\{u_n\}$ or $\{x_n\}$ or $\{f(n)\}$, where u_n or x_n or $f(n)$ is the nth term of the respective sequence. Thus the domain of a real sequence is the set of natural numbers and the range is a sub set of the set of reals.

2.**Constant sequence:**
A sequence $\{u_n\}$ is said to be a constant sequence if $u_n = a \ \forall \ n \in \ \mathbf{N}$ (constant). Thus the range set of a constant sequence is a singleton set.

3. **Equality of two sequences:**
Two sequences $\{u_n\}$ and $\{v_n\}$ are said to be equal if $u_n = v_n \forall \ n \ \in \ \mathbf{N}$.

4. **Bounded Sequence:**
A real sequence $\{u_n\}$ is said to be bounded above if $\exists$ a real number B s.t.
$u_n \ \leq B \ \forall \ n \ \in \ \mathbf{N}$. B is said to be an upper bound of the sequence.
A real sequence $\{u_n\}$ is said to be bounded below if $\exists$ a real number b s.t.
$b \ \leq u_n \ \forall \ n \ \in \ \mathbf{N}$. b is said to be a lower bound of the sequence.
A real sequence $\{u_n\}$ is said to be bounded if it is bounded above as well as bounded below, i.e. $\exists$ two real numbers B and b s.t. $b \ \leq \ u_n \ \leq B \ \forall \ n \ \in \ \mathbf{N}$.
If the sequence is bounded the the range set is bounded and conversely.

5. **Least upper and greatest lower bounds:**
If a sequence $\{u_n\}$ is bounded above, then the range set of it is also bounded above and a non empty sub set of **R**. Thus by the supremum property of **R**, the range set has the least upper bound (l.u.b), which is also called the least upper bound of the sequence $\{u_n\}$ and is denoted by $Sup\{u_n\}$.
Thus the least upper bound of a real sequence $\{u_n\}$ is a real number M satisfying the following conditions:
(i) $u_n \ \leq \ M \ \forall \ n \in \ \mathbf{N}$. and
(ii) for each pre-assigned $\epsilon > 0 \ \exists \ k \ \in \mathbf{N}$ s.t. $M - \epsilon \ < \ u_k$.
Similarly, if a sequence $\{u_n\}$ is bounded below, then the range set of it is also bounded below and a non empty sub set of **R**. Thus by the infimum property of **R**, the range

set has the greatest lower bound(g.l.b), which is also called the greatest lower bound of the sequence $\{u_n\}$ and is denoted by $inf\{u_n\}$.
Thus the greatest lower bound of a real sequence $\{u_n\}$ is a real number m satisfying the following conditions:
(i) $m \leq u_n \; \forall \; n \in$ **N**. and
(ii) for each pre-assigned $\epsilon > 0 \; \exists \; k \; \in \mathbf{N}$ s.t. $u_k < m + \epsilon$.
It is important to note here that for a real sequence $\{u_n\}$ which is unbounded above, we have $Sup\{u_n\} = \infty$ and a sequence $\{v_n\}$which is unbounded below, we have $inf\{v_n\} = -\infty$.

6. **Limit of a sequence:**
A real number u is said to be the limit of a sequence $\{u_n\}$ if corresponding to a pre-assigned $\epsilon > 0 \; \exists \; a, \; k \in \mathbf{N}$ (k depends on ϵ) s.t. $|u_n - u| < \epsilon \; \forall \; n \geq k$ i.e. $u - \epsilon < u_n < u + \epsilon \; \forall \; n \geq k$, i.e. all the elements of the sequence, excepting the first $(k-1)$ at most, lie in the $\epsilon-$nbd of u.

7. **Convergent Sequence:**
A real sequence $\{u_n\}$ is said to be converge to a real number u if for any given $\epsilon > 0 \; \exists \; a, \; k \in \mathbf{N}$ s.t. $|u_n - u| < \epsilon \; \forall \; n \geq k$, i.e. $u - \epsilon < u_n < u + \epsilon \; \forall \; n \geq k$, i.e. if the sequence has a limit in **R**. Then we write $lim u_n = u$.

8. **Prove that a convergent sequence is bounded but not conversely.**
Proof: Let $\{u_n\}$ be a convergent sequence and u be its limit. Then for any given $\epsilon > 0 \; \exists \; a, \; k \in \mathbf{N}$ s.t. $|u_n - u| < \epsilon \; \forall \; n \geq k$, i.e. $u - \epsilon < u_n < u + \epsilon \; \forall \; n \geq k$. Let $b = min\{u_1, u_2, ...u_{k-1}, u - \epsilon\}$ and $B = max\{u_1, u_2, ...u_{k-1}, u + \epsilon\}\}$. Then we have $b \leq u_n \leq B \; \forall \; n \in$ **N**. Hence the sequence $\{u_n\}$ is bounded.
Now establish that the converse of the theorem is not true, let us consider the sequence $\{(-1)^n\}$. Since the range set of this sequence is the set $\{-1, \; 1\}$, which is bounded and hence the sequence is bounded. The sequence is not convergent as it has two limit points -1 and 1.

9. **Divergent sequence :**
A sequence $\{u_n\}$ is said to be diverges to ∞ if for every real number $G > 0$ (however large) $\exists$ a $k \in \mathbf{N}$ s.t. $G < u_n \forall \; n \geq k$. If the sequence diverges to infinity the we write $lim \; u_n = \infty$.
A sequence $\{u_n\}$ is said to be diverges to $-\infty$ if for every real number $G > 0$ (however large) $\exists$ a $k \in \mathbf{N}$ s.t. $u_n < -G \forall \; n \geq k$. If the sequence diverges to $-\infty$ then we write $lim \; u_n = -\infty$.

A real sequence is said to be properly divergent if it is either diverges to ∞ or diverges to $-\infty$.

10.**Oscillatory sequence:**
A sequence $\{u_n\}$ is said to be an oscillatory sequence if it is neither convergent nor properly divergent.
Since every convergent sequence is bounded but the converse is not true. A bounded sequence that is not convergent is said to be an oscillatory sequence of finite oscillation.
An unbounded sequence that is not properly divergent is said to be an oscillatory sequence of infinite oscillation.
An oscillatory sequence is also called improperly divergent sequence.

11.**Null sequence:** A sequence $\{u_n\}$ is said to be a Null sequence if $lim u_n = 0$